DIRAC EQUATION IN RELATIVISTIC QUANTUM MECHANICS: SUMMARY

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.21.

At this point, it's useful to summarize the Dirac equation and its solutions as found in relativistic quantum theory. First, the theory that we've studied so far is *not* field theory; the solutions ψ or $|\psi\rangle$ are generalizations of the solutions of the Schrödinger equation, in which the position **x** of a particle is an operator and is itself a function of time. In field theory, **x** and t are placed on an equal footing and both serve as nothing more than labels of points in spacetime.

The Dirac equation was postulated to have the same form as the Schrödinger equation, that is

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \tag{1}$$

To make the equation consistent with relativity, it was postulated that for a free particle, the Hamiltonian had the form

$$H = \alpha\cdot\mathbf{p}+\beta m \tag{2}$$

Requiring that the square of this Hamiltonian satisfies the relativistic energy-momentum relation $E^2 = p^2+m^2$ leads to conditions on the four matrices α and β:

$$\begin{aligned} \{\alpha_i,\alpha_j\} &= \{\alpha_i,\beta\} = 0 \text{ if } i\ne j & (3)\\ \alpha_i^2=\beta^2 &= I & (4)\end{aligned}$$

Working with the Dirac equation is easier if we introduce the gamma matrices which allows the Dirac equation to be written in the compact form

$$(i\gamma^\mu\partial_\mu - m)|\psi\rangle = 0 \tag{5}$$

Remember that the γ^μ constitute four 4×4 matrices, and that $|\psi\rangle$ is a 4-d column vector, so this simple form of the Dirac equation actually represents four coupled PDEs in the four spacetime variables.

At this point, we are given one set of solutions to 5, written in the form

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{bmatrix}e^{-ipx} \equiv u_1 e^{-ipx} \tag{6}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\end{bmatrix}e^{-ipx} \equiv u_2 e^{-ipx} \tag{7}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{bmatrix}e^{ipx} \equiv v_2 e^{ipx} \tag{8}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\\0\\1\end{bmatrix}e^{ipx} \equiv v_1 e^{ipx} \tag{9}$$

Here the u_r and v_r are the 4-d spinor components, and the exponentials $e^{\pm ipx}$ contain the spacetime dependence. [Again, remember that these solutions are only for a free particle; there is no potential term and hence no interactions.] The inner products of the spinors satisfy several normalization and orthogonality conditions, such as

$$\begin{aligned}
u_r^\dagger(\mathbf{p})\,u_r(\mathbf{p}) &= v_r^\dagger(\mathbf{p})\,v_r(\mathbf{p}) = \frac{E}{m};\ r=1,2 &(10)\\
u_1^\dagger(\mathbf{p})\,u_2(-\mathbf{p}) &= 0 &(11)\\
u_1^\dagger(\mathbf{p})\,v_2(-\mathbf{p}) &= 0 &(12)
\end{aligned}$$

The overall solutions satisfy the orthogonality conditions

$$\left\langle \psi^{(r)}(\mathbf{p})\middle|\psi^{(s)}(\mathbf{p})\right\rangle = \frac{EV}{m}\delta_{rs} \tag{13}$$

where V is the volume containing the particle.
The adjoint solutions are defined by

$$\left\langle\bar{\psi}^{(n)}\right| \equiv \left\langle\psi^{(n)}\right|\gamma^0 \tag{14}$$

These solutions satisfy the adjoint Dirac equation

$$i\partial_\mu \left\langle \bar{\psi}\right| \gamma^\mu + m \left\langle \bar{\psi}\right| = 0 \tag{15}$$

If we defined a probability current by

$$j^\mu \equiv \left\langle \bar{\psi} \right| \gamma^\mu \left| \psi \right\rangle \tag{16}$$

we find that it satisfies the conservation law

$$\partial_\mu j^\mu = 0 \tag{17}$$

Using the $\mu = 0$ component of j^μ as a probability density ρ leads to positive probabilities for both particles and antiparticles. However, antiparticles still end up with negative energy.

The spin and helicity properties of the Dirac equation are summarized in Klauber's Wholeness Chart 4-1.

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