DIRAC EQUATION IN QUANTUM FIELD THEORY: LAGRANGIAN DENSITY

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.22.

Our treatment of the Dirac equation so far has been restricted to relativistic quantum theory, and has not touched on field theory. To make the transition to field theory, we start with the same equation, namely

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0 \tag{1}$$

except that now ψ is interpreted as a field, rather than a quantum state. Since this equation is mathematically identical to the Dirac equation that we solved for quantum states, its solutions are the same as well

$$\psi^{(1)}=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}1\\0\\\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{1}e^{-ipx} \tag{2}$$

$$\psi^{(2)}=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}0\\1\\\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\end{array}\right]e^{-ipx}\equiv u_{2}e^{-ipx} \tag{3}$$

$$\psi^{(3)}=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}\frac{p^{3}}{E+m}\\\frac{p^{1}+ip^{2}}{E+m}\\1\\0\end{array}\right]e^{ipx}\equiv v_{2}e^{ipx} \tag{4}$$

$$\psi^{(4)}=\sqrt{\frac{E+m}{2m}}\left[\begin{array}{c}\frac{p^{1}-ip^{2}}{E+m}\\-\frac{p^{3}}{E+m}\\0\\1\end{array}\right]e^{ipx}\equiv v_{1}e^{ipx} \tag{5}$$

The adjoint Dirac equation also remains the same form

$$i\partial_\mu\bar{\psi}\gamma^\mu+m\bar{\psi}=0 \tag{6}$$

and has solutions

$$\bar{\psi}^{(1)} = u_1^\dagger\gamma^0e^{ipx}\equiv\bar{u}_1e^{ipx} \tag{7}$$

$$\bar{\psi}^{(2)} = u_2^\dagger\gamma^0e^{ipx}\equiv\bar{u}_2e^{ipx} \tag{8}$$

$$\bar{\psi}^{(3)} = v_2^\dagger\gamma^0e^{ipx}\equiv\bar{v}_2e^{-ipx} \tag{9}$$

$$\bar{\psi}^{(4)} = v_1^\dagger\gamma^0e^{ipx}\equiv\bar{v}_1e^{-ipx} \tag{10}$$

The general solutions for ψ and $\bar{\psi}$ are linear combinations of these solutions, which are written as

$$\psi=\sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx}+d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right] \tag{11}$$

$$\equiv\psi^++\psi^- \tag{12}$$

$$\bar{\psi}=\sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx}+c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx}\right] \tag{13}$$

$$\equiv\bar{\psi}^++\bar{\psi}^- \tag{14}$$

where V is the volume containing the particles.

Again, these solutions are mathematically equivalent to the solutions of the quantum state Dirac equation, except that the coefficients c_r and d_r now turn out to be operators rather than just numbers.

To proceed further with field theory, we need the Lagrangian and Hamiltonian densities for the Dirac equation. As with the Klein-Gordon equation, we'll pull the Lagrangian density out of thin air (Klauber says it is obtained by 'trial and error'):

$$\mathscr{L}_0^{1/2}=\bar{\psi}\left(i\gamma^\mu\partial_\mu-m\right)\psi \tag{15}$$

The $\frac{1}{2}$ superscript on $\mathscr{L}$ signifies that we're dealing with spin-$\frac{1}{2}$ particles (and is *not* a square root!), while the subscript 0 indicates that we're dealing with free particles. [It's worth remembering at this point that this is a matrix equation; $\bar{\psi}$ is a 4-d row vector, ψ is a 4-d column vector, γ^μ is a 4×4 matrix and m is multiplied by the 4×4 identity matrix. After multiplying all these matrices together, though, the Lagrangian density is a scalar.] We can verify that this Lagrangian density does give the Dirac equation and its adjoint by plugging it into the Euler-Lagrange equation. We have

$$\frac{d}{dx^{\mu}}\left(\frac{\partial\mathscr{L}}{\partial\phi_{,\mu}^{n}}\right)-\frac{\partial\mathscr{L}}{\partial\phi^{n}}=0 \tag{16}$$

where the index n indicates which field we're considering. Here, we have two fields: $\phi^1=\bar{\psi}$ and $\phi^2=\psi$. For ϕ^1 we have

$$\frac{d}{dx^{\mu}}\left(\bar{\psi}i\gamma^{\mu}\right)-\left(-\bar{\psi}m\right) = 0 \tag{17}$$

$$i\partial_{\mu}\bar{\psi}\gamma^{\mu}+m\bar{\psi} = 0 \tag{18}$$

This is adjoint Dirac equation 6 above.

For ϕ^2 we get, since there are no derivatives of $\bar{\psi}$ in the Lagrangian 15,

$$\frac{d}{dx^{\mu}}\left(\frac{\partial\mathscr{L}}{\partial\bar{\psi}_{,\mu}}\right) = 0 \tag{19}$$

$$\frac{\partial\mathscr{L}}{\partial\bar{\psi}} = \left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0 \tag{20}$$

which is just the original Dirac equation 1.

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