HAMILTONIAN DENSITY FOR THE DIRAC EQUATION

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.23.

The Lagrangian density which gives the Dirac equation is

$$\mathcal{L}_0^{1/2}=\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi \tag{1}$$

We can follow the same procedure that we used for scalar fields to derive the conjugate momenta for the Dirac spin-$\frac{1}{2}$ field. For the conjugate momenta we have from 1

$$\begin{aligned}
\pi^{1/2} &\equiv \frac{\partial\mathcal{L}_0^{1/2}}{\partial\psi_{,0}} && (2)\\
&= i\bar{\psi}\gamma^0 && (3)\\
&= i\psi^{\dagger}\gamma^0\gamma^0 && (4)\\
&= i\psi^{\dagger} && (5)
\end{aligned}$$

where we've used the definition of the adjoint field $\bar{\psi}\equiv\psi^{\dagger}\gamma^0$.

Because the Lagrangian is not symmetric with respect to ψ and $\bar{\psi}$ (it contains derivatives of ψ but not of $\bar{\psi}$), the adjoint momentum turns out to be zero:

$$\bar{\pi}^{1/2}\equiv\frac{\partial\mathcal{L}_0^{1/2}}{\partial\bar{\psi}_{,0}}=0 \tag{6}$$

From the conjugate momenta and the Lagrangian, we can derive the Hamiltonian density:

$$\begin{aligned}
\mathcal{H}_0^{1/2} &= \pi_r^{1/2}\phi_{,0}^r && (7)\\
&= \pi^{1/2}\dot{\psi}+\bar{\pi}^{1/2}\dot{\bar{\psi}}-\mathcal{L}_0^{1/2} && (8)\\
&= i\psi^{\dagger}\dot{\psi}+0-\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi && (9)\\
&= i\bar{\psi}\gamma^0\dot{\psi}-i\bar{\psi}\gamma^0\dot{\psi}-i\bar{\psi}\gamma^j\partial_j\psi+m\bar{\psi}\psi && (10)\\
&= -i\bar{\psi}\gamma^j\partial_j\psi+m\bar{\psi}\psi && (11)
\end{aligned}$$

where in the last 2 lines, the j index is summed over spatial coordinates only.

The general solution of the Dirac equation for discrete momenta is

$$\psi = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx}+d_r^{\dagger}(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{12}$$

$$\bar{\psi} = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,e^{-ipx}+c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,e^{ipx}\right] \tag{13}$$

We can find the total Hamiltonian $H_0^{1/2}$ by substituting these into 11 and integrating over space:

$$H_0^{1/2} = \int \mathcal{H}_0^{1/2} d^3x \tag{14}$$

$$= \int\left[-i\bar{\psi}\gamma^j\partial_j\psi+m\bar{\psi}\psi\right]d^3x \tag{15}$$

As you might guess, this is a messy operation, and Klauber goes through (most) of the gory details in his section 4.4.1. Basically, we substitute 12 and 13 into 15, using different indices for the sums in each case ($r,\mathbf{p}$ for ψ and $s,\mathbf{p}'$ for $\bar{\psi}$). We then use the fact that the integral of $e^{\pm i\mathbf{p}\cdot\mathbf{x}}$ over all space gives zero unless $\mathbf{p}=0$ (because we're assuming that the volume V is such that an integral number of wavelengths fits exactly within its boundaries). This allows us to collapse the double sum over $\mathbf{p}$ and $\mathbf{p}'$ to a single sum over $\mathbf{p}$, although we must still retain the double sum over the spins r and s. We get a set of four integrals resulting from the derivative term in 15 and another four integrals resulting from the mass term in 15. We need to add each integral in the first set of four to the corresponding integral in the second set. We'll consider the sum of the fourth integral from each set here (the derivation is similar to that done by Klauber in his Box 4-3, where he adds the first integral from each set). The fourth integral from the derivative term is (implied sum over $j=1,2,3$)

$$I_{d4} \equiv -\int\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}c_r^{\dagger}(-\mathbf{p})\,\bar{u}_r(-\mathbf{p})\,\gamma^j p^j v_s(\mathbf{p})\,d_s^{\dagger}(\mathbf{p})\,e^{2iE_{\mathbf{p}}t}d^3x \tag{16}$$

The fourth integral from the mass term is

$$I_{m4} \equiv \int\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}c_r^{\dagger}(-\mathbf{p})\,\bar{u}_r(-\mathbf{p})\,mv_s(\mathbf{p})\,d_s^{\dagger}(\mathbf{p})\,e^{2iE_{\mathbf{p}}t}d^3x \tag{17}$$

The only differences between the two integrals are the minus sign in I_{d4} and the replacement of $\gamma^j p^j$ in I_{d4} by m in I_{m4}.

[In Klauber's Box 4-3, he uses a symbol with a slash through it to indicate a sum of that symbol multiplied by γ^μ. Unfortunately, the wordpress.com Latex server doesn't support symbols with a slash through them, so I'll need to resort to the longhand notation.]

Suppose we have a single eigensolution which is a single term from the last term in 12, that is (no sum over r; we can take r to be either spin):

$$\psi=\sqrt{\frac{m}{VE_{\mathbf{p}}}}d_r^\dagger(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx} \tag{18}$$

Since this is a solution of the Dirac equation we must have

$$\begin{aligned}\left(i\gamma^\mu\partial_\mu-m\right)\psi &= 0 &(19)\\ \sqrt{\frac{m}{VE_{\mathbf{p}}}}d_r^\dagger(\mathbf{p})\,e^{ipx}\left(-\gamma^\mu p_\mu-m\right)v_r(\mathbf{p}) &= 0 &(20)\end{aligned}$$

Remember that this is a matrix equation, and note that the terms $\sqrt{\frac{m}{VE_{\mathbf{p}}}}d_r^\dagger(\mathbf{p})\,e^{ipx}$ are not matrices, nor are they zero, so we must have

$$\left(\gamma^\mu p_\mu+m\right)v_r(\mathbf{p})=0 \tag{21}$$

Returning to 16, we can convert the integrand using the inner product of the spinor terms

$$u_r^\dagger(-\mathbf{p})\,v_s(\mathbf{p})=0 \tag{22}$$

From the definition of the adjoint spinor $\bar{u}_r$ and the identity $\gamma^0\gamma^0=1$, we have

$$\begin{aligned}u_r^\dagger(-\mathbf{p})\,v_s(\mathbf{p}) &= u_r^\dagger(-\mathbf{p})\,\gamma^0\gamma^0 v_s(\mathbf{p}) &(23)\\ &= \bar{u}_r(-\mathbf{p})\,\gamma^0 v_s(\mathbf{p}) &(24)\\ &= 0 &(25)\end{aligned}$$

Since this term is zero, we can multiply it by anything without changing it, so we multiply it by p_0 to get

$$u_r^\dagger(-\mathbf{p})\,v_s(\mathbf{p})=\bar{u}_r(-\mathbf{p})\,\gamma^0 p_0 v_s(\mathbf{p})=0 \tag{26}$$

The middle part of 16 can be written as

$$\bar{u}_r(-\mathbf{p})\gamma^j p^j v_s(\mathbf{p}) = \bar{u}_r(-\mathbf{p})\gamma^j p_j v_s(\mathbf{p}) \tag{27}$$
$$= -\bar{u}_r(-\mathbf{p})\gamma^0 p_0 v_s(\mathbf{p}) + \bar{u}_r(-\mathbf{p})\gamma^j p_j v_s(\mathbf{p}) \tag{28}$$
$$= -\bar{u}_r(-\mathbf{p})\gamma^\mu p_\mu v_s(\mathbf{p}) \tag{29}$$
$$= \bar{u}_r(-\mathbf{p}) m v_s(\mathbf{p}) \tag{30}$$

where we subtracted 26 (which is zero, so it doesn't change anything) in the second line and used 21 to get the last line. Putting this final result into 16 we get

$$I_{d4} = -\int \sum_{r,s,\mathbf{p}} \frac{m}{VE_{\mathbf{p}}} c_r^\dagger(-\mathbf{p})\bar{u}_r(-\mathbf{p}) m v_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) e^{2iE_{\mathbf{p}}t} d^3 = -I_{m4} \tag{31}$$

so $I_{d4} + I_{m4} = 0$ and the two integrals cancel each other.

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