

HAMILTONIAN FOR THE DIRAC EQUATION

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.24.

Here we continue with the derivation of the total Dirac Hamiltonian. We've seen that the Hamiltonian density is given as

$$(1) \quad \mathcal{H}_0^{1/2} = -i\bar{\psi}\gamma^j\partial_j\psi + m\bar{\psi}\psi$$

where the general solutions of the Dirac equation are given by

$$(2) \quad \psi = \sum_{r=1}^2 \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[c_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right]$$

$$(3) \quad \bar{\psi} = \sum_{r=1}^2 \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ipx} + c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{ipx} \right]$$

The full Hamiltonian is the integral of the density over all space, that is

$$(4) \quad H_0^{1/2} = \int [-i\bar{\psi}\gamma^j\partial_j\psi + m\bar{\psi}\psi] d^3x$$

From section 4.4.1 in Klauber's book and the previous post, we see that four of the terms in the expansion of this integral (after substituting in 2 and 3) cancel out, leaving us with (implied sum over $j = 1, 2, 3$ in the first two lines):

$$(5) \quad \begin{aligned} H_0^{1/2} = & - \int \sum_{r,s,\mathbf{p}} \frac{m}{VE_{\mathbf{p}}} d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) \gamma^j p^j v_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) d^3x + \\ & \int \sum_{r,s,\mathbf{p}} \frac{m}{VE_{\mathbf{p}}} c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) \gamma^j p^j u_s(\mathbf{p}) c_s(\mathbf{p}) d^3x + \\ & \int \sum_{r,s,\mathbf{p}} \frac{m}{VE_{\mathbf{p}}} d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) m v_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) d^3x + \\ & \int \sum_{r,s,\mathbf{p}} \frac{m}{VE_{\mathbf{p}}} c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) m u_s(\mathbf{p}) c_s(\mathbf{p}) d^3x \end{aligned}$$

Klauber shows that the sum of the first and third lines gives us

$$(6) \quad \sum_{r,\mathbf{p}} E_{\mathbf{p}} d^{\dagger}(\mathbf{p}) d_r(\mathbf{p}) - 1 \equiv \sum_{r,\mathbf{p}} E_{\mathbf{p}} \bar{N}_r(\mathbf{p}) - 1$$

By following similar steps to Klauber's equations 4-70 to 4.73, we can work out the sum of the second and fourth lines, as follows. We can use the inner products of the spinors:

$$(7) \quad u_r^{\dagger}(\mathbf{p}) u_s(\mathbf{p}) = v_r^{\dagger}(\mathbf{p}) v_s(\mathbf{p}) = \frac{E_{\mathbf{p}}}{m} \delta_{rs} = \frac{p_0}{m} \delta_{rs}$$

The adjoint spinors are defined as

$$(8) \quad \bar{u}_r = u_r^{\dagger} \gamma^0$$

so combined with the identity $\gamma^0 \gamma^0 = 1$ we have

$$(9) \quad u_r^{\dagger}(\mathbf{p}) u_s(\mathbf{p}) = \bar{u}_r(\mathbf{p}) \gamma^0 u_s(\mathbf{p})$$

$$(10) \quad \bar{u}_r(\mathbf{p}) \gamma^0 p_0 u_s(\mathbf{p}) = \frac{(p_0)^2}{m} \delta_{rs}$$

$$(11) \quad = \frac{E_{\mathbf{p}}^2}{m} \delta_{rs}$$

The sum of the second and fourth lines of 5 can then be written as

$$(12) \quad \int \sum_{r,s,\mathbf{p}} \frac{m}{V E_{\mathbf{p}}} c_r^{\dagger}(\mathbf{p}) \bar{u}_r(\mathbf{p}) [-\gamma^j p_j + m - \gamma^0 p_0 + \gamma^0 p_0] u_s(\mathbf{p}) c_s(\mathbf{p}) d^3x =$$

$$(13) \quad \int \frac{d^3x}{V} \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} c_r^{\dagger}(\mathbf{p}) \left[\bar{u}_r(\mathbf{p}) (-\gamma^{\mu} p_{\mu} + m) u_r(\mathbf{p}) + \frac{E_{\mathbf{p}}^2}{m} \right] c_r(\mathbf{p})$$

Using Klauber's equation B4-3.3:

$$(14) \quad (\gamma^{\mu} p_{\mu} - m) u_r(\mathbf{p}) = 0$$

and the fact that

$$(15) \quad \int d^3x = V$$

that is, the integral over the entire volume is just the volume V itself, we see that 13 reduces to

$$(16) \quad \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} c_r^\dagger(\mathbf{p}) \frac{E_{\mathbf{p}}^2}{m} c_r(\mathbf{p}) = \sum_{r,\mathbf{p}} E_{\mathbf{p}} c_r^\dagger(\mathbf{p}) c_r(\mathbf{p})$$

$$(17) \quad \equiv \sum_{r,\mathbf{p}} E_{\mathbf{p}} N_r(\mathbf{p})$$

Combining this result with 6 we get the final Hamiltonian as

$$(18) \quad H_0^{1/2} = \sum_{r,\mathbf{p}} E_{\mathbf{p}} \left(N_r(\mathbf{p}) - \frac{1}{2} + \bar{N}_r(\mathbf{p}) - \frac{1}{2} \right)$$

The operators are interpreted as number operators so that $N_r(\mathbf{p})$ is the operator whose eigenvalue is the number of particles with spin r and momentum \mathbf{p} , and $\bar{N}_r(\mathbf{p})$ is the operator whose eigenvalue is the number of antiparticles with spin r and momentum \mathbf{p} . The two $\frac{1}{2}$ terms give rise to an infinite negative vacuum energy.

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