ANTICOMMUTATORS, CREATION AND ANNIHILATION OPERATORS IN THE DIRAC EQUATION

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Reference: References: Robert D. Klauber, *Student Friendly Quantum Field Theory*, (Sandtrove Press, 2013) - Chapter 4, Problem 4.25.

In the case of the scalar field, the second quantization postulate is that the Poisson brackets of classical field theory translate into commutators in quantum field theory. For the Dirac equation, however, the four solution vectors contain 4-d spinors, for which there are no analogies in classical theory. As a result, there are no Poisson brackets for spinor fields, so we can't apply second quantization in the same way as for the Klein-Gordon field. In fact, it turns out that the coefficients in the general solution of the Dirac equation in quantum field theory obey anticommutation relations. The general solutions are

$$\psi = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx}+d_r^{\dagger}(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{1}$$

$$\equiv \psi^{+}+\psi^{-} \tag{2}$$

$$\bar{\psi} = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,e^{-ipx}+c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,e^{ipx}\right] \tag{3}$$

$$\equiv \bar{\psi}^{+}+\bar{\psi}^{-} \tag{4}$$

where the coefficients $c_r(\mathbf{p})$ and $d_r(\mathbf{p})$ (and their Hermitian conjugates) are operators rather than numbers. It is these operators that obey the anticommutation relations, which as far as I can tell, are just postulates of the theory. The anticommutation relations are

$$\left[c_r(\mathbf{p}),c_s^{\dagger}(\mathbf{p}')\right]_{+}=\left[d_r(\mathbf{p}),d_s^{\dagger}(\mathbf{p}')\right]_{+}=\delta_{rs}\delta_{\mathbf{pp}'} \tag{5}$$

[Note that some books use braces $\{\}$ to indicate anticommutators rather than the $[]_{+}$ notation used by Klauber.] All other anticommutators are taken to be zero. One consequence of this latter property is that all the operators anticommute with themselves, that is

$$\left[c_r(\mathbf{p}), c_r(\mathbf{p})\right]_+ = \left[c_r^\dagger(\mathbf{p}), c_r^\dagger(\mathbf{p})\right]_+ = 0 \tag{6}$$

$$\left[d_r(\mathbf{p}), d_r(\mathbf{p})\right]_+ = \left[d_r^\dagger(\mathbf{p}), d_r^\dagger(\mathbf{p})\right]_+ = 0 \tag{7}$$

Or, in other words

$$\left[c_r(\mathbf{p})\right]^2 = \left[c_r^\dagger(\mathbf{p})\right]^2 = \left[d_r(\mathbf{p})\right]^2 = \left[d_r^\dagger(\mathbf{p})\right]^2 = 0 \tag{8}$$

Just as in the case of the scalar field, these operators prove to be creation and annihilation operators. As shown by Klauber in his section 4.6.1, if we operate on a state containing a single particle in the state $\left|\psi_{r,\mathbf{p}}\right\rangle$ with the operator $c_r(\mathbf{p})$ we get an eigenstate of the number operator $N_r(\mathbf{p})$ with eigenvalue 0:

$$N_r(\mathbf{p})\, c_r(\mathbf{p}) \left|\psi_{r,\mathbf{p}}\right\rangle = 0 \times c_r(\mathbf{p}) \left|\psi_{r,\mathbf{p}}\right\rangle \tag{9}$$

That is, the state $c_r(\mathbf{p})\left|\psi_{r,\mathbf{p}}\right\rangle$ is the vacuum state $|0\rangle$. Similarly, the operator $c_r^\dagger(\mathbf{p})$ creates a particle with spin r and momentum $\mathbf{p}$ when it operates on the vacuum state (or on any state that does not already contain a particle with spin r and momentum $\mathbf{p}$):

$$N_r(\mathbf{p})\, c_r^\dagger(\mathbf{p}) |0\rangle = 1 \times c_r^\dagger(\mathbf{p}) |0\rangle \tag{10}$$

so that

$$c_r^\dagger(\mathbf{p}) |0\rangle = \left|\psi_{r,\mathbf{p}}\right\rangle \tag{11}$$

However, if we try to create another particle of the same spin and momentum in the same state, we get zero because of 8:

$$\left[c_r^\dagger(\mathbf{p})\right]^2 |0\rangle = 0 \tag{12}$$

Note the subtle distinction between 9 and 12. In the former, we destroy an existing particle thus producing the vacuum state, which is real physical state containing no particles. In the latter, we actually annihilate the vacuum state by attempting to create two identical particles in it, producing zero, that is, nothing at all (not even the vacuum). This is the theory's way of telling us that it is physically impossible to create two identical fermions, an effect we encountered previously in non-relativistic quantum theory.

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