CREATION AND ANNIHILATION OPERATORS: COMMUTATORS AND ANTICOMMUTATORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 1, Problems 1.1 - 1.2.

As a bit of background to the quantum field theoretic use of creation and annihilation operators we'll look again at the harmonic oscillator. The creation and annihilation operators (called raising and lowering operators by Griffiths) are defined in terms of the position and momentum operators as

$$a^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}}\left[-ip+m\omega x\right] \tag{1}$$

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left[ip+m\omega x\right] \tag{2}$$

From the commutator $[x,p]=i\hbar$ we can work out

$$\left[a,a^{\dagger}\right] = \frac{1}{2\hbar m\omega}\left(-im\omega\left[x,p\right]\right) \tag{3}$$

$$= 1 \tag{4}$$

The annihilation operator a acting on the vacuum or ground state $|0\rangle$ gives 0, and the creation operator $a^{\dagger}$ produces a state $a^{\dagger}|0\rangle = |1\rangle$ with energy eigenvalue $\frac{3}{2}\hbar\omega$. Successive applications of $a^{\dagger}$ produce states with higher energy, where each quantum of energy is $\hbar\omega$.

Normalization. Given that the ground state is normalized so that $\langle 0|0\rangle = 1$, we can find the factor required to normalize higher states so that $\langle n|n\rangle = 1$. Consider $n=2$. We have

$$a^{\dagger}a^{\dagger}|0\rangle = A|2\rangle \tag{5}$$

where A is to be determined. We have

$$\left\langle 0\left|aaa^{\dagger}a^{\dagger}\right|0\right\rangle = \left\langle 0\left|a\left(1+a^{\dagger}a\right)a^{\dagger}\right|0\right\rangle \tag{6}$$
$$= \left\langle 0\left|aa^{\dagger}\right|0\right\rangle + \left\langle 0\left|aa^{\dagger}aa^{\dagger}\right|0\right\rangle \tag{7}$$
$$= \left\langle 0\left|\left(1+a^{\dagger}a\right)\right|0\right\rangle + \left\langle 0\left|aa^{\dagger}\left(1+a^{\dagger}a\right)\right|0\right\rangle \tag{8}$$
$$= \langle 0|0\rangle + \left\langle 0\left|aa^{\dagger}\right|0\right\rangle \tag{9}$$
$$= \langle 0|0\rangle + \left\langle 0\left|\left(1+a^{\dagger}a\right)\right|0\right\rangle \tag{10}$$
$$= \langle 0|0\rangle + \langle 0|0\rangle \tag{11}$$
$$= 2 \tag{12}$$
$$= \frac{1}{A^2} \tag{13}$$
$$A = \frac{1}{\sqrt{2}} \tag{14}$$

For $n = 3$ we get $\left\langle 0\left|aaaa^{\dagger}a^{\dagger}a^{\dagger}\right|0\right\rangle$. We need to commute each a through the $a^{\dagger}$ operators to its right. The first a will generate the factor $\left(1+a^{\dagger}a\right)$ 3 times as it commutes with each $a^{\dagger}$ operator. Each of these terms will be $\left\langle 0\left|aaa^{\dagger}a^{\dagger}\right|0\right\rangle$ and we already know that this term produces a factor of 2. Therefore

$$\left\langle 0\left|aaaa^{\dagger}a^{\dagger}a^{\dagger}\right|0\right\rangle = 3\times 2 = 6 \tag{15}$$

We can extend this result to the general case:

$$\left\langle 0\left|a^{n}\left(a^{\dagger}\right)^{n}\right|0\right\rangle = n! \tag{16}$$

The normalization must then be

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^{n}|0\rangle \tag{17}$$

Number operator. We've met the number operator N in the field case, but there is an analogous operator for the harmonic oscillator. We have

$$N \equiv a^{\dagger}a \tag{18}$$

As with the field case, we can work out its commutators:

$$\left[N,a^\dagger\right] = a^\dagger a a^\dagger - a^\dagger a^\dagger a \tag{19}$$
$$= a^\dagger a^\dagger a + a^\dagger - a^\dagger a^\dagger a \tag{20}$$
$$= a^\dagger \tag{21}$$
$$[N,a] = a^\dagger a a - a a^\dagger a \tag{22}$$
$$= a^\dagger a a - a + a^\dagger a a \tag{23}$$
$$= -a \tag{24}$$

Applying this to $|n\rangle$ we get

$$N|n\rangle = \frac{1}{\sqrt{n!}} N\left(a^\dagger\right)^n |0\rangle \tag{25}$$

We get

$$N\left(a^\dagger\right)^n = \left[a^\dagger + a^\dagger N\right]\left(a^\dagger\right)^{n-1} \tag{26}$$
$$= \left(a^\dagger\right)^n + \left(a^\dagger\right)^2 (1+N)\left(a^\dagger\right)^{n-2} \tag{27}$$
$$= \ldots \tag{28}$$
$$= n\left(a^\dagger\right)^n + \left(a^\dagger\right)^n N \tag{29}$$
$$= n\left(a^\dagger\right)^n + \left(a^\dagger\right)^n a^\dagger a \tag{30}$$

When operating on $|0\rangle$, the last term gives 0, so

$$N|n\rangle = \frac{n}{\sqrt{n!}}\left(a^\dagger\right)^n |0\rangle \tag{31}$$

Multiple oscillators. If we now have a system of N non-interacting harmonic oscillators with equal masses and frequencies ω_i, $i = 1, \ldots, N$, the Hamiltonian is

$$H = \frac{1}{2m}\sum_i \left(p_i^2 + m^2\omega_i^2 x_i^2\right) \tag{32}$$

Since the oscillators are not coupled, the creation and annihilation operators for different operators all commute, so that

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \tag{33}$$

so the normalized state where oscillator i is in the n_ith excited state is

$$|n_1 n_2 \ldots n_N\rangle = \prod_{i=1}^{N} \frac{\left(a_i^\dagger\right)^{n_i}}{\sqrt{n_i!}} |0\rangle \tag{34}$$

The number operator in this case is

$$\mathscr{N} = \sum_{i=1}^{N} \left(a_i^\dagger a_i\right) \tag{35}$$

This works because the commutation relation 33 allows each term $a_i^\dagger a_i$ in the sum to pick out the number of quanta of oscillator i.

Anticommutators. Now suppose that instead of the commutation relations 33 we have *anticommutation* relations as follows:

$$\left\{a_i, a_j^\dagger\right\} \equiv a_i a_j + a_j a_i = \delta_{ij} \tag{36}$$

$$\left\{a_i^\dagger, a_j^\dagger\right\} = \left\{a_i, a_j\right\} = 0 \tag{37}$$

If we start with the vacuum state $|0\rangle$ and require $a_i^\dagger |0\rangle = |0 \ldots 1_i \ldots 0\rangle$ (that is, $a_i^\dagger$ creates one quantum in category i), then if we try to create another quantum in the same state, we get

$$\left\langle 0 \left| a_i a_i a_i^\dagger a_i^\dagger \right| 0 \right\rangle = \left\langle 0 \left| a_i \left(1 - a_i^\dagger a_i\right) a_i^\dagger \right| 0 \right\rangle \tag{38}$$

$$= \left\langle 0 \left| a_i a_i^\dagger \right| 0 \right\rangle - \left\langle 0 \left| a_i a_i^\dagger a_i a_i^\dagger \right| 0 \right\rangle \tag{39}$$

$$= \left\langle 0 \left| a_i a_i^\dagger \right| 0 \right\rangle - \left\langle 0 \left| a_i a_i^\dagger \left(1 - a_i^\dagger a_i\right) \right| 0 \right\rangle \tag{40}$$

$$= \left\langle 0 \left| a_i a_i^\dagger \right| 0 \right\rangle - \left\langle 0 \left| a_i a_i^\dagger \right| 0 \right\rangle + \left\langle 0 \left| a_i a_i^\dagger a_i^\dagger a_i \right| 0 \right\rangle \tag{41}$$

$$= 0 \tag{42}$$

Thus, attempting to create two quanta in the same state produces zero, so at most one quantum can occupy each state. The commutator case 33 thus behaves like bosons and the anticommutator case like fermions.