LORENTZ TRANSFORMATION FOR INFINITESIMAL RELATIVE VELOCITY

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 1, Problems 1.5 - 1.6.

In special relativity, Lahiri & Pal use the opposite metric to the one we've been using so far, in that $g_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$, that is, the time component is positive and the spatial components are negative. With this definition, lowering or raising the 0 index of a tensor has no effect on the sign, while lowering or raising index 1, 2 or 3 changes the sign.

With the usual spacetime four-vector

$$x^{\mu} \equiv \left(x^{0},x^{i}\right) = (ct,\mathbf{x}) \tag{1}$$

the lowered version is

$$x_{\mu} = g_{\mu\nu}x^{\nu} = (ct,-\mathbf{x}) \tag{2}$$

Under a Lorentz transformation, the x^{μ} transform as

$$x^{\prime\mu} = \Lambda_{\nu}^{\mu}x^{\nu} \tag{3}$$

The transformation for x_{μ} is therefore

$$\begin{aligned}
x_{\mu}^{\prime} &= g_{\mu\nu}x^{\prime\nu} && (4)\\
&= g_{\mu\nu}\Lambda_{\sigma}^{\nu}x^{\sigma} && (5)\\
&= \Lambda_{\mu\sigma}g^{\sigma\rho}x_{\rho} && (6)\\
&= \Lambda_{\mu}^{\rho}x_{\rho} && (7)
\end{aligned}$$

The matrix Λ_{μ}^{ρ} is the original matrix Λ_{ρ}^{μ} with the first index lowered and second raised. If $\mu=\rho=0$ or if both μ and ρ are spatial indices, the matrix element remains unchanged: $\Lambda_{\mu}^{\rho} = \Lambda_{\rho}^{\mu}$. If, however, exactly one index is zero (with the other index being spatial), the element changes sign: $\Lambda_{\mu}^{\rho} = -\Lambda_{\rho}^{\mu}$.

Infinitesimal relative velocity. In the standard case where the primed frame is moving relative to the unprimed frame at speed v along the x axis, the Lorentz transformations are

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{8}$$

$$x' = \gamma(x - vt) \tag{9}$$

$$y' = y \tag{10}$$

$$z' = z \tag{11}$$

with

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{12}$$

If $\frac{v}{c}$ is very small we can expand these equations to first order in $\beta \equiv \frac{v}{c}$. To this order

$$\gamma = 1 + \frac{\beta^2}{2} + \ldots \tag{13}$$

$$\approx 1 \tag{14}$$

and

$$ct' = ct - x\beta \tag{15}$$

$$x' = x - ct\beta \tag{16}$$

so

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Lowering the first index we get

$$\Lambda_{\mu\nu} = g_{\mu\rho}\Lambda^{\rho}_{\nu} \tag{18}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 1 & -\beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{20}$$

We can write this as the sum of $g_{\mu\nu}$ and an antisymmetric matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$:

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu} \tag{21}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$