

LORENTZ TRANSFORMATION FOR INFINITESIMAL RELATIVE VELOCITY

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 1, Problems 1.5 - 1.6.

In special relativity, Lahiri & Pal use the opposite metric to the one we've been using so far, in that $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, that is, the time component is positive and the spatial components are negative. With this definition, lowering or raising the 0 index of a tensor has no effect on the sign, while lowering or raising index 1, 2 or 3 changes the sign.

With the usual spacetime four-vector

$$x^\mu \equiv (x^0, x^i) = (ct, \mathbf{x}) \quad (1)$$

the lowered version is

$$x_\mu = g_{\mu\nu}x^\nu = (ct, -\mathbf{x}) \quad (2)$$

Under a Lorentz transformation, the x^μ transform as

$$x'^\mu = \Lambda^\mu_\nu x^\nu \quad (3)$$

The transformation for x_μ is therefore

$$x'_\mu = g_{\mu\nu}x'^\nu \quad (4)$$

$$= g_{\mu\nu}\Lambda^\nu_\sigma x^\sigma \quad (5)$$

$$= \Lambda_{\mu\sigma}g^{\sigma\rho}x_\rho \quad (6)$$

$$= \Lambda_\mu^\rho x_\rho \quad (7)$$

The matrix Λ_μ^ρ is the original matrix Λ^μ_ρ with the first index lowered and second raised. If $\mu = \rho = 0$ or if both μ and ρ are spatial indices, the matrix element remains unchanged: $\Lambda_\mu^\rho = \Lambda^\mu_\rho$. If, however, exactly one index is zero (with the other index being spatial), the element changes sign: $\Lambda_\mu^\rho = -\Lambda^\mu_\rho$.

Infinitesimal relative velocity. In the standard case where the primed frame is moving relative to the unprimed frame at speed v along the x axis, the Lorentz transformations are

$$t' = \gamma \left(t - \frac{vx}{c^2} \right) \quad (8)$$

$$x' = \gamma(x - vt) \quad (9)$$

$$y' = y \quad (10)$$

$$z' = z \quad (11)$$

with

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \quad (12)$$

If $\frac{v}{c}$ is very small we can expand these equations to first order in $\beta \equiv \frac{v}{c}$. To this order

$$\gamma = 1 + \frac{\beta^2}{2} + \dots \quad (13)$$

$$\approx 1 \quad (14)$$

and

$$ct' = ct - x\beta \quad (15)$$

$$x' = x - ct\beta \quad (16)$$

so

$$\Lambda_{\mathbf{v}}^{\mu} = \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (17)$$

Lowering the first index we get

$$\Lambda_{\mu\nu} = g_{\mu\rho}\Lambda_{\nu}^{\rho} \quad (18)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

$$= \begin{bmatrix} 1 & -\beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (20)$$

We can write this as the sum of $g_{\mu\nu}$ and an antisymmetric matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$:

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu} \quad (21)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (22)$$