MAXWELL'S EQUATIONS USING THE ELECTROMAGNETIC FIELD TENSOR

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 1, Problem 1.7.

We can summarize the electromagnetic field in tensor form by means of the field tensor

$$F_{\mu\nu}=\left[\begin{array}{cccc}0 & -E_{x} & -E_{y} & -E_{z}\\ E_{x} & 0 & B_{z} & -B_{y}\\ E_{y} & -B_{z} & 0 & B_{x}\\ E_{z} & B_{y} & -B_{x} & 0\end{array}\right] \tag{1}$$

[This tensor is written using relativistic units with $c=1$ so that $\mathbf{E}$ and $\mathbf{B}$ have the same dimensions.]

We've already seen that the pair of homogeneous Maxwell's equations can be written in terms of this tensor as follows:

$$\partial_{\mu}F_{\nu\sigma}+\partial_{\sigma}F_{\mu\nu}+\partial_{\nu}F_{\sigma\mu}=0 \tag{2}$$

With the usual ordering of coordinates $\left(x^{0},x^{1},x^{2},x^{3}\right)=(t,x,y,z)$, if we set $\mu=2$, $\nu=1$ and $\sigma=3$ we get

$$\begin{aligned}-\partial_{y}B_{y}-\partial_{z}B_{z}-\partial_{x}B_{x} &= 0 &\quad(3)\\ \nabla\cdot\mathbf{B} &= 0 &\quad(4)\end{aligned}$$

Selecting $\mu=2$, $\nu=1$ and $\sigma=0$ gives

$$\partial_{y}E_{x}-\partial_{t}B_{z}-\partial_{x}E_{y}=0 \tag{5}$$

This is the z component of

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{6}$$

We can get the x component by choosing $\mu=0$, $\nu=2$ and $\sigma=3$:

$$\partial_t B_x - \partial_z E_y + \partial_y E_z = 0 \tag{7}$$

The y component comes from $\mu = 0$, $\nu = 1$ and $\sigma = 3$:

$$-\partial_t B_y - \partial_z E_x + \partial_x E_z = 0 \tag{8}$$

The two inhomogenous Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{9}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{10}$$

$$= \frac{1}{c^2}\left(\frac{\mathbf{J}}{\varepsilon_0} + \frac{\partial \mathbf{E}}{\partial t}\right) \tag{11}$$

$$= \frac{\mathbf{J}}{\varepsilon_0} + \frac{\partial \mathbf{E}}{\partial t} \tag{12}$$

where we used $\mu_0 \varepsilon_0 = 1/c^2$ and the last line uses relativistic units with $c = 1$.

We need to introduce the four-current to put these in four-vector form. This is

$$J^\mu = [\rho, \mathbf{J}] \tag{13}$$

where ρ is the charge density and $\mathbf{J}$ is the three-current. Then if we look at Gauss's law 9 we see that this can be written as

$$\partial_\nu F^{0\nu} = \frac{J^0}{\varepsilon_0} \tag{14}$$

where $F^{\mu\nu}$ is the raised version of the tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{15}$$

If we generalize this formula we get

$$\partial_\nu F^{\mu\nu} = \frac{J^\mu}{\varepsilon_0} \tag{16}$$

For $\mu = 1$ we get

$$-\partial_t E_x + \partial_y B_z - \partial_z B_y = \frac{J^x}{\varepsilon_0} \tag{17}$$

This is the x component of 12. Choosing $\mu = 2$ and $\mu = 3$ give the y and z components respectively.

From our examination of the electromagnetic tensor, we saw the four-vector form of the Lorentz force law for a charge q:

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \tag{18}$$

where τ is the proper time, p^μ is the four-momentum and u_ν is the four-velocity.

To summarize, Maxwell's equations can be written as

$$\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} = 0 \tag{19}$$

$$\partial_\nu F^{\mu\nu} = \frac{J^\mu}{\varepsilon_0} \tag{20}$$

The Lorentz force law can be written as

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \tag{21}$$

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