

MAXWELL'S EQUATIONS USING THE ELECTROMAGNETIC FIELD TENSOR

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 1, Problem 1.7.

We can summarize the electromagnetic field in tensor form by means of the field tensor

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \quad (1)$$

[This tensor is written using relativistic units with $c = 1$ so that \mathbf{E} and \mathbf{B} have the same dimensions.]

We've already seen that the pair of homogeneous Maxwell's equations can be written in terms of this tensor as follows:

$$\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} = 0 \quad (2)$$

With the usual ordering of coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, if we set $\mu = 2$, $\nu = 1$ and $\sigma = 3$ we get

$$-\partial_y B_y - \partial_z B_z - \partial_x B_x = 0 \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (4)$$

Selecting $\mu = 2$, $\nu = 1$ and $\sigma = 0$ gives

$$\partial_y E_x - \partial_t B_z - \partial_x E_y = 0 \quad (5)$$

This is the z component of

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (6)$$

We can get the x component by choosing $\mu = 0$, $\nu = 2$ and $\sigma = 3$:

$$\partial_t B_x - \partial_z E_y + \partial_y E_z = 0 \quad (7)$$

The y component comes from $\mu = 0$, $\nu = 1$ and $\sigma = 3$:

$$-\partial_t B_y - \partial_z E_x + \partial_x E_z = 0 \quad (8)$$

The two inhomogenous Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (9)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (10)$$

$$= \frac{1}{c^2} \left(\frac{\mathbf{J}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t} \right) \quad (11)$$

$$= \frac{\mathbf{J}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t} \quad (12)$$

where we used $\mu_0 \epsilon_0 = 1/c^2$ and the last line uses relativistic units with $c = 1$.

We need to introduce the four-current to put these in four-vector form. This is

$$J^\mu = [\rho, \mathbf{J}] \quad (13)$$

where ρ is the charge density and \mathbf{J} is the three-current. Then if we look at Gauss's law 9 we see that this can be written as

$$\partial_\nu F^{0\nu} = \frac{J^0}{\epsilon_0} \quad (14)$$

where $F^{\mu\nu}$ is the raised version of the tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (15)$$

If we generalize this formula we get

$$\partial_\nu F^{\mu\nu} = \frac{J^\mu}{\epsilon_0} \quad (16)$$

For $\mu = 1$ we get

$$-\partial_t E_x + \partial_y B_z - \partial_z B_y = \frac{J^x}{\epsilon_0} \quad (17)$$

This is the x component of 12. Choosing $\mu = 2$ and $\mu = 3$ give the y and z components respectively.

From our examination of the electromagnetic tensor, we saw the four-vector form of the Lorentz force law for a charge q :

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \quad (18)$$

where τ is the proper time, p^μ is the four-momentum and u_ν is the four-velocity.

To summarize, Maxwell's equations can be written as

$$\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} = 0 \quad (19)$$

$$\partial_\nu F^{\mu\nu} = \frac{J^\mu}{\epsilon_0} \quad (20)$$

The Lorentz force law can be written as

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \quad (21)$$

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