LAGRANGIAN FOR INHOMOGENEOUS MAXWELL'S EQUATIONS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.1.

The Euler-Lagrange equations for a classical field are

$$\frac{\partial\mathcal{L}}{\partial\phi^r}-\frac{\partial}{\partial q^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^r_{,\mu}}\right)=0 \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density, ϕ^r is the rth field, q^μ are the generalized coordinates and the notation $\phi_{,\mu}\equiv\frac{\partial\phi}{\partial q^\mu}$. As an example of how these equations can give rise to physical equations that are more familiar, we'll look at the Lagrangian (I'll leave off the 'density' to save space, but when talking about fields instead of particles, we'll always mean Lagrangian *density*) for the electromagnetic field. At this point, we won't worry about where this Lagrangian comes from; we'll just state it as

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-j_\mu A^\mu \tag{2}$$

where the electromagnetic field tensor is (This definition is taken from Moore's book on general relativity and is actually the negative of Lahiri & Pal's definition in their equation 8.6. We don't actually use this form of the tensor in what follows so it doesn't affect the result, but it's important to realize that different authors use different definitions.)

$$F_{\mu\nu}=\left[\begin{array}{cccc}0 & -E_x & -E_y & -E_z\\ E_x & 0 & B_z & -B_y\\ E_y & -B_z & 0 & B_x\\ E_z & B_y & -B_x & 0\end{array}\right] \tag{3}$$

the four-current is

$$j^\mu=[\rho,\mathbf{J}] \tag{4}$$

and A^μ is the four-potential

$$A^\mu\equiv[V,\mathbf{A}] \tag{5}$$

In applying 1, we take the fields ϕ^r to be the potentials A^μ. What do the Euler-Lagrange equations give us for these fields?

The first term in 1 is just

$$\frac{\partial\mathcal{L}}{\partial A^\mu} = -j_\mu \tag{6}$$

The second term is a bit more involved. First, we need to write $F_{\mu\nu}$ in terms of the four-potential, which is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

In calculating the derivatives, we need to be careful to keep track of the positions (up or down) of the various indices. The ϕ^r in 1 are represented by A^μ with raised indices. The derivative becomes

$$\phi^r_{,\alpha} = A^\mu_{,\alpha} = \frac{\partial A^\mu}{\partial q^\alpha} = \partial_\alpha A^\mu \tag{8}$$

That is, the index on the derivative is lowered and the index on the field is raised. To take the derivatives of $\mathcal{L}$ we need to express $F^{\mu\nu}F_{\mu\nu}$ in this form, so we get

$$F^{\mu\nu}F_{\mu\nu} = \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{9}$$

$$= \left(g^{\mu\pi}\partial_\pi A^\nu - g^{\nu\pi}\partial_\pi A^\mu\right)\left(g_{\nu\rho}\partial_\mu A^\rho - g_{\mu\rho}\partial_\nu A^\rho\right) \tag{10}$$

where $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\left(+1,-1,-1,-1\right)$ using Lahiri and Pal's convention. Now we can calculate the inner derivative in the second term of 1. First, we'll consider the field $\phi^r = A^0$. Using the product rule we get (I'll use α for the coordinate index to avoid confusion with μ which is used to denote the component of A^μ; remember also that $g^{\mu\nu}$ is diagonal, and there is no implied sum on α):

$$\begin{aligned}
-4\frac{\partial\mathcal{L}}{\partial A^0_{,\alpha}} = {} & g^{\alpha\alpha}\left[g_{00}\partial_\alpha A^0 - g_{\alpha\alpha}\partial_0 A^\alpha\right] - g^{\alpha\alpha}\left[g_{\alpha\alpha}\partial_0 A^\alpha - g_{00}\partial_\alpha A^0\right] + \\
& g_{00}\left[g^{\alpha\alpha}\partial_\alpha A^0 - g^{00}\partial_0 A^\alpha\right] - g_{00}\left[g^{00}\partial_0 A^\alpha - g^{\alpha\alpha}\partial_\alpha A^0\right] \qquad (11)\\
= {} & 4\left[g^{\alpha\alpha}\partial_\alpha A^0 - \partial_0 A^\alpha\right] \qquad (12)
\end{aligned}$$

When $\alpha = 0$, the RHS is zero, and for $\alpha = 1,2,3$, $g^{\alpha\alpha} = -1$ so

$$\frac{\partial\mathcal{L}}{\partial A^0_{,\alpha}} = \begin{cases} 0 & \alpha = 0 \\ \partial_i A^0 + \partial_0 A^i & i = 1,2,3 \end{cases} \tag{13}$$

Plugging this into 1 we get, together with 6

$$-j^0 = \sum_{i=1}^{3} \partial_i \left(\partial_i A^0 + \partial_0 A^i\right) \tag{14}$$

$$-\rho = \nabla^2 V + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}\right) \tag{15}$$

This is the potential equivalent of the Maxwell equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ (the system of units used in Lahiri & Pal takes $\epsilon_0 = 1$).

Now look at the field A^1. We have

$$-4\frac{\partial \mathcal{L}}{\partial A^1_{,\alpha}} = g^{\alpha\alpha}\left[g_{11}\partial_\alpha A^1 - g_{\alpha\alpha}\partial_1 A^\alpha\right] - g^{\alpha\alpha}\left[g_{\alpha\alpha}\partial_1 A^\alpha - g_{11}\partial_\alpha A^1\right] +$$

$$g_{11}\left[g^{\alpha\alpha}\partial_\alpha A^1 - g^{11}\partial_1 A^\alpha\right] - g_{11}\left[g^{11}\partial_1 A^\alpha - g^{\alpha\alpha}\partial_\alpha A^1\right] \tag{16}$$

$$= -4\left[g^{\alpha\alpha}\partial_\alpha A^1 + \partial_1 A^\alpha\right] \tag{17}$$

We get

$$\frac{\partial \mathcal{L}}{\partial A^1_{,\alpha}} = \begin{cases} \partial_0 A^1 + \partial_1 A^0 & \alpha = 0 \\ -\partial_i A^1 + \partial_1 A^i & i = 1,2,3 \end{cases} \tag{18}$$

Plugging this into 1 we have

$$\partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial A^1_{,\alpha}}\right) = \frac{\partial^2 A^1}{\partial t^2} + \partial_x\left(\partial_t A^0\right) - \nabla^2 A^1 + \partial_x\left(\nabla \cdot \mathbf{A}\right) \tag{19}$$

$$= \frac{\partial^2 A_x}{\partial t^2} + \left[\nabla\left(\partial_t V\right)\right]_x - \nabla^2 A_x + \left[\nabla\left(\nabla \cdot \mathbf{A}\right)\right]_x \tag{20}$$

Combining this with 6 gives

$$\frac{\partial^2 A_x}{\partial t^2} + \left[\nabla\left(\partial_t V\right)\right]_x - \nabla^2 A_x + \left[\nabla\left(\nabla \cdot \mathbf{A}\right)\right]_x = -j_1 = +j^1 = j_x \tag{21}$$

Doing the same calculation for A^2 and A^3 yields the y and z components, so we get

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla\left(\partial_t V\right) - \nabla^2 \mathbf{A} + \nabla\left(\nabla \cdot \mathbf{A}\right) = \mathbf{J} \tag{22}$$

which is the other inhomogeneous Maxwell equation in potential form.

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