EULER-LAGRANGE EQUATIONS: A COUPLE OF EXAMPLES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problems 2.2 - 2.3.

The Euler-Lagrange equations for a classical field are

$$\frac{\partial\mathscr{L}}{\partial\phi^r}-\frac{\partial}{\partial q^\mu}\left(\frac{\partial\mathscr{L}}{\partial\phi^r_{,\mu}}\right)=0 \tag{1}$$

where $\mathscr{L}$ is the Lagrangian density, ϕ^r is the rth field, q^μ are the generalized coordinates and the notation $\phi_{,\mu}\equiv\frac{\partial\phi}{\partial q^\mu}$. Here are a couple of examples of using this equation.

Example 1. We have a real scalar field ϕ with Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)-\frac{1}{2}m^2\phi^2-V(\phi) &(2)\\ &= \frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right)-\frac{1}{2}m^2\phi^2-V(\phi) &(3)\end{aligned}$$

In this case, there is only a single field ϕ, so the Euler-Lagrange equation is found as follows.

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\phi} &= -m^2\phi-\frac{\partial V}{\partial\phi} &(4)\\ \frac{\partial\mathscr{L}}{\partial\phi_{,\mu}} &= g^{\mu\mu}\left(\partial_\mu\phi\right)\text{ (no sum)} &(5)\\ \frac{\partial}{\partial q^\mu}\left(\frac{\partial\mathscr{L}}{\partial\phi^r_{,\mu}}\right) &= \sum_{\mu=0}^{3}g^{\mu\mu}\left(\partial^2_\mu\phi\right) &(6)\\ &= \frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi &(7)\\ &= \Box\phi &(8)\end{aligned}$$

where $\Box$ is the d'Alembertian operator

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{9}$$

Thus the Euler-Lagrange equation is

$$\left(\Box + m^2\right)\phi = -\frac{\partial V}{\partial \phi} \tag{10}$$

Example 2. Now suppose the field ϕ is complex, and the Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^\dagger \left(\partial^\mu \phi\right) - \frac{1}{2}m^2 \phi^\dagger \phi - V\left(\phi^\dagger \phi\right) \tag{11}$$

$$= \frac{1}{2} g^{\mu\nu}\left(\partial_\mu \phi\right)^\dagger \left(\partial_\nu \phi\right) - \frac{1}{2}m^2 \phi^\dagger \phi - V\left(\phi^\dagger \phi\right) \tag{12}$$

Here there are actually two fields (the real and imaginary parts of ϕ). We can regard ϕ and its hermitian conjugate $\phi^\dagger$ as independent variables in calculating the Euler-Lagrange equations. We can do this, even though once we know ϕ we also know $\phi^\dagger$, for much the same reason that we can regard ϕ and $\partial_\mu \phi$ as independent variables. That is, we need to find ϕ such that the action is an extremum, and the action is determined by the integral of $\mathscr{L}$, which depends on both ϕ and $\phi^\dagger$.

Applying 1 with $\phi^r = \phi$ we get

$$\frac{\partial \mathscr{L}}{\partial \phi} = -m^2 \phi^\dagger - \frac{\partial V}{\partial \phi} \tag{13}$$

$$\frac{\partial \mathscr{L}}{\partial \phi_{,\mu}} = g^{\mu\mu}\left(\partial_\mu \phi\right)^\dagger \text{ (no sum)} \tag{14}$$

$$\frac{\partial}{\partial q^\mu}\left(\frac{\partial \mathscr{L}}{\partial \phi^r_{,\mu}}\right) = \sum_{\mu=0}^{3} g^{\mu\mu}\left(\partial^2_\mu \phi\right)^\dagger \tag{15}$$

$$= \frac{\partial^2 \phi^\dagger}{\partial t^2} - \nabla^2 \phi^\dagger \tag{16}$$

$$= \Box \phi^\dagger \tag{17}$$

Therefore the Euler-Lagrange equation is

$$\left(\Box + m^2\right)\phi^\dagger = -\frac{\partial V}{\partial \phi} \tag{18}$$

[Technical note for those who know a bit of complex variable theory: The derivative $\frac{\partial V}{\partial \phi}$ is with respect to a *complex* quantity ϕ, so in order for this derivative to exist over the complex plane, the potential V would have to satisfy the Cauchy-Riemann equations, which would imply that V itself must have non-zero real and imaginary parts. A physical potential function, however, must be real, so it would seem that this derivative must be defined in some other way. I'm not sure how to interpret this equation.]

If we take $\phi^r = \phi^\dagger$, we just get the conjugate of 18, so there is only one independent Euler-Lagrange equation here.

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