

EULER-LAGRANGE EQUATIONS: A COUPLE OF EXAMPLES

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problems 2.2 - 2.3.

The Euler-Lagrange equations for a classical field are

$$\frac{\partial \mathcal{L}}{\partial \phi^r} - \frac{\partial}{\partial q^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}^r} \right) = 0 \quad (1)$$

where \mathcal{L} is the Lagrangian density, ϕ^r is the r th field, q^μ are the generalized coordinates and the notation $\phi_{,\mu} \equiv \frac{\partial \phi}{\partial q^\mu}$. Here are a couple of examples of using this equation.

Example 1. We have a real scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - V(\phi) \quad (2)$$

$$= \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - \frac{1}{2} m^2 \phi^2 - V(\phi) \quad (3)$$

In this case, there is only a single field ϕ , so the Euler-Lagrange equation is found as follows.

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi - \frac{\partial V}{\partial \phi} \quad (4)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = g^{\mu\mu} (\partial_\mu \phi) \text{ (no sum)} \quad (5)$$

$$\frac{\partial}{\partial q^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) = \sum_{\mu=0}^3 g^{\mu\mu} (\partial_\mu^2 \phi) \quad (6)$$

$$= \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi \quad (7)$$

$$= \square \phi \quad (8)$$

where \square is the d'Alembertian operator

$$\square \equiv \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (9)$$

Thus the Euler-Lagrange equation is

$$(\square + m^2) \phi = -\frac{\partial V}{\partial \phi} \quad (10)$$

Example 2. Now suppose the field ϕ is complex, and the Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi) \quad (11)$$

$$= g^{\mu\nu} (\partial_\mu \phi)^\dagger (\partial_\nu \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi) \quad (12)$$

Here there are actually two fields (the real and imaginary parts of ϕ). We can regard ϕ and its hermitian conjugate ϕ^\dagger as independent variables in calculating the Euler-Lagrange equations. We can do this, even though once we know ϕ we also know ϕ^\dagger , for much the same reason that we can regard ϕ and $\partial_\mu \phi$ as independent variables. That is, we need to find ϕ such that the action is an extremum, and the action is determined by the integral of \mathcal{L} , which depends on both ϕ and ϕ^\dagger .

Applying 1 with $\phi^r = \phi$ we get

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^\dagger - \frac{\partial V}{\partial \phi} \quad (13)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = g^{\mu\mu} (\partial_\mu \phi)^\dagger \text{ (no sum)} \quad (14)$$

$$\frac{\partial}{\partial q^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) = \sum_{\mu=0}^3 g^{\mu\mu} (\partial_\mu^2 \phi)^\dagger \quad (15)$$

$$= \frac{\partial^2 \phi^\dagger}{\partial t^2} - \nabla^2 \phi^\dagger \quad (16)$$

$$= \square \phi^\dagger \quad (17)$$

Therefore the Euler-Lagrange equation is

$$(\square + m^2) \phi^\dagger = -\frac{\partial V}{\partial \phi} \quad (18)$$

[Technical note for those who know a bit of complex variable theory: The derivative $\frac{\partial V}{\partial \phi}$ is with respect to a *complex* quantity ϕ , so in order for this derivative to exist over the complex plane, the potential V would have to satisfy the Cauchy-Riemann equations, which would imply that V itself

must have non-zero real and imaginary parts. A physical potential function, however, must be real, so it would seem that this derivative must be defined in some other way. I'm not sure how to interpret this equation.]

If we take $\phi^r = \phi^\dagger$, we just get the conjugate of 18, so there is only one independent Euler-Lagrange equation here.

PINGBACKS

Pingback: Klein-Gordon equation: plane wave solutions

Pingback: Hamiltonian for complex scalar field

Pingback: Stress-energy tensor: 3 examples

Pingback: Noether's theorem - internal symmetry of complex scalar field

Pingback: Noether's theorem - internal symmetry and scaled spacetime

Pingback: Natural Units