CANONICAL MOMENTA FOR ELECTROMAGNETIC FIELDS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problems 2.4 - 2.5.

The Lagrangian density for the classical electromagnetic field is

$$\mathscr{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-j_{\mu}A^{\mu} \tag{1}$$

where the electromagnetic field tensor is given in terms of the four-potential, which is

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu} \tag{2}$$

If we treat the four components A^{μ} as the fields in the field theory, then we can calculate the canonical momentum associated with each field. Using Lahiri & Pal's notation, this is given by

$$\Pi_{A}\equiv\frac{\delta L}{\delta\dot{\Phi}^{A}} \tag{3}$$

where Π_A is the momentum associated with field Φ_A, and a dot indicates a time derivative, that is, a derivative with respect to x_0. The δ means we are taking a functional derivative, since L, the total Lagrangian ($L=\int d^3x\,\mathscr{L}$) is a functional which depends on the function $\mathscr{L}$, which is, in turn, a function of the fields and their derivatives.

To calculate 3, we need to know the functional derivatives of the fields and their derivatives with respect to each other. This is done by analogy with the generalized coordinates and momenta from classical Lagrangian theory, and the result is given in L&P's equation 2.27:

$$\frac{\delta\dot{\Phi}^{B}(t,\mathbf{y})}{\delta\dot{\Phi}^{A}(t,\mathbf{x})}=\delta_{A}^{B}\delta^{3}(\mathbf{x}-\mathbf{y}) \tag{4}$$

The functional derivatives of all fields and their spatial derivatives with respect to any $\dot{\Phi}^{A}$ are taken to be zero.

To work out 3 for the Lagrangian 1, we first write out the expression in full:

$$L = \int d^3x \mathscr{L} \tag{5}$$

$$= -\int d^3x \left[\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + j_\mu A^\mu\right] \tag{6}$$

We now need to calculate the derivative 3, for which we can use the product rule, since functional derivatives obey the same rules are ordinary derivatives. To get the signs right, we need to keep careful track of the location (up or down) of the various indexes. If we're calculating Π_A (that is, with a lower index), then we need to take the derivative with respect to $\partial^0\Phi^A$ (that is, the field has an upper index; the location of the 0 index on the ∂ doesn't matter since it's the same up or down). Therefore, we'd like all the terms in 6 to have upper indexes, and for that we can use the metric tensor. Thus we have

$$L = -\int d^3x \left[\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(g_{\mu\alpha}g_{\nu\beta}\partial^\alpha A^\beta - g_{\mu\beta}g_{\nu\alpha}\partial^\alpha A^\beta\right) + j_\mu A^\mu\right] \tag{7}$$

Now we can find the derivative $\Pi_\rho = \frac{\delta L}{\delta \dot{A}^\rho} = \frac{\delta L}{\delta\left(\partial^0 A^\rho\right)}$. The derivative consists of four contributions: the derivative of each of the two terms in the first parentheses multiplied by the second parentheses, and the first parentheses multiplied by the derivative of each othe two terms in the second parentheses. We illustrate a couple of these derivatives as the other two are essentially the same.

First, consider the derivative of $\partial^\mu A^\nu$ multiplied by $\left(g_{\mu\alpha}g_{\nu\beta}\partial^\alpha A^\beta - g_{\mu\beta}g_{\nu\alpha}\partial^\alpha A^\beta\right)$. We have

$$\frac{\delta\left(\partial^\mu A^\nu(t,\mathbf{y})\right)}{\delta\left(\partial^0 A^\rho(t,\mathbf{x})\right)} = \delta_0^\mu \delta_\rho^\nu \delta^3(\mathbf{x}-\mathbf{y}) \tag{8}$$

Multiplying this by the second parentheses and doing the implied sums over repeated indexes we have

$$\delta_0^\mu \delta_\rho^\nu \delta^3(\mathbf{x}-\mathbf{y})\left(g_{\mu\alpha}g_{\nu\beta}\partial^\alpha A^\beta - g_{\mu\beta}g_{\nu\alpha}\partial^\alpha A^\beta\right) = \left(\partial^0 g_{\rho\beta}A^\beta - g_{\rho\alpha}\partial^\alpha A^0\right)\delta^3(\mathbf{x}-\mathbf{y}) \tag{9}$$

$$= \left(\dot{A}_\rho - \partial_\rho A_0\right)\delta^3(\mathbf{x}-\mathbf{y}) \tag{10}$$

where we've used the fact that a 0 index is the same in the up or down position. When we do the spatial integral in 7, the spatial delta function disappears. Restoring the $-\frac{1}{4}$ means this first of four terms gives us

$$-\frac{1}{4}\left(\dot{A}_{\rho}-\partial_{\rho}A_{0}\right) \tag{11}$$

Working out the derivative of the second term in the first parentheses in 7 gives us the same thing, except with the sign reversed. However, the minus sign before the second term in the first parentheses in 7 means that both terms give us the same result, which is 11

Now consider the first parentheses multiplied by the derivative of the first term in the second parentheses. First, work out the derivative of this term:

$$\frac{\delta\left(g_{\mu\alpha}g_{\nu\beta}\partial^{\alpha}A^{\beta}\left(t,\mathbf{y}\right)\right)}{\delta\left(\partial^{0}A^{\rho}\left(t,\mathbf{x}\right)\right)}=g_{\mu\alpha}g_{\nu\beta}\delta_{0}^{\alpha}\delta_{\rho}^{\beta}\delta^{3}\left(\mathbf{x}-\mathbf{y}\right) \tag{12}$$

Multiplying in the first parentheses we get

$$\left(\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}\right)g_{\mu\alpha}g_{\nu\beta}\delta_{0}^{\alpha}\delta_{\rho}^{\beta}\delta^{3}\left(\mathbf{x}-\mathbf{y}\right)=g_{\nu\rho}\left(\partial^{0}A^{\nu}-\partial^{\nu}A^{0}\right)\delta^{3}\left(\mathbf{x}-\mathbf{y}\right) \tag{13}$$

$$=\left(\dot{A}_{\rho}-\partial_{\rho}A_{0}\right)\delta^{3}\left(\mathbf{x}-\mathbf{y}\right) \tag{14}$$

This is the same result that we got when working out the first term in 10. Finally, we can work out the first parentheses multiplied by the derivative of the second term in the second parentheses and we get the same answer (with sign reversed). Integrating over space gets rid of the spatial delta function, as before.

Thus the final result for Π_{ρ} is four times the result 11, or

$$\Pi_{\rho} = \partial_{\rho}A_{0}-\dot{A}_{\rho} \tag{15}$$

$$= \partial_{\rho}A_{0}-\partial_{0}A_{\rho} \tag{16}$$

$$= F_{\rho 0} \tag{17}$$

Notice that $\Pi_{0}=0$, so the canonical momentum has only 3 non-zero components. As a result, it's not possible to invert these equations to write $\dot{A}_{\rho}$ in terms of Π_{ρ} since, in general, A_{0} is not zero. However, if we invoke the gauge in which $A_{0}=0$, then we still get $\Pi_{0}=0$, but now the spatial momenta become

This gauge does not appear to be either the Coulomb or Lorenz gauge.

$$\Pi_{i}=-\dot{A}_{i} \tag{18}$$

or, flipping the index:

$$\Pi^i = \dot{A}^i \tag{19}$$

So in this case the spatial canonical momenta are just the time derivatives of the corresponding field components.

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