ONE-DIMENSIONAL FIELD (DISPLACEMENT OF A STRING)

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.7.

Here we'll look at the field theory for the displacement of a string of length l with fixed ends (such as a violin string). The Lagrangian (total, not just the density) is

$$L=\int_0^l dx\left[\left(\frac{\partial u}{\partial t}\right)^2-c^2\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{1}$$

If this string is plucked, then it will vibrate so that the displacement u varies with position x and time t. We can write the displacement as a Fourier series:

$$u(x,t)=\sum_{k=1}^{\infty}q_k(t)\sin\left(\frac{\omega_k x}{c}\right) \tag{2}$$

where

$$\omega_k=\frac{\pi kc}{l} \tag{3}$$

Here k is an integer and q_k can be thought of as a parameter which gives the contribution to the overall displacement of the pure sine wave with mode ω_k. The q_k are the generalized coordinates in the problem, so we can calculate the conjugate momentum using the standard formula

$$p_k=\frac{\partial L}{\partial\dot{q}_k} \tag{4}$$

In our case, the Lagrangian is an integral over the square of an infinite series, so the situation might look hopeless. However, it's not as bad as it looks, due to the fact that sine terms in the Fourier series are orthogonal functions when integrated over the interval $[0,l]$. That is

$$\int_0^l \sin\left(\frac{\omega_a x}{c}\right)\sin\left(\frac{\omega_b x}{c}\right)dx = \int_0^l \sin\left(\frac{\pi a x}{l}\right)\sin\left(\frac{\pi b x}{l}\right)dx \tag{5}$$

$$= \frac{l}{2}\delta_{ab} \tag{6}$$

From 2

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} \dot{q}_k(t)\sin\left(\frac{\omega_k x}{c}\right) \tag{7}$$

We can ignore the second term in the integrand of 1 when calculating p_k since it doesn't contain $\dot{q}_k$. We therefore have

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{8}$$

$$= 2\int_0^l dx \frac{\partial u}{\partial t}\sin\left(\frac{\omega_k x}{c}\right) \tag{9}$$

$$= 2\int_0^l dx \left(\sum_{n=1}^{\infty} \dot{q}_n(t)\sin\left(\frac{\omega_n x}{c}\right)\right)\sin\left(\frac{\omega_k x}{c}\right) \tag{10}$$

Using 6, we see that only one of the terms (when $n = k$) in the sum will contribute to the integral, so we get

$$p_k = 2\frac{l}{2}\dot{q}_k = l\dot{q}_k \tag{11}$$

The full Lagrangian in 1 can be worked out in a similar way. To integrate the square of 7 we use 6, so the only terms that contribute to the integral are the terms involving $\dot{q}_k^2$:

$$\int_0^l dx \left(\frac{\partial u}{\partial t}\right)^2 = \sum_{k=1}^{\infty}\int_0^l \dot{q}_k^2 \sin^2\frac{\omega_k x}{c} dx \tag{12}$$

$$= \frac{l}{2}\sum_{k=1}^{\infty}\dot{q}_k^2 \tag{13}$$

$$= \frac{1}{2l}\sum_{k=1}^{\infty}p_k^2 \tag{14}$$

The cosine is also orthogonal when integrated over $[0, l]$ so we have

$$\int_0^l \cos\left(\frac{\omega_a x}{c}\right)\cos\left(\frac{\omega_b x}{c}\right)dx = \int_0^l \cos\left(\frac{\pi a x}{l}\right)\cos\left(\frac{\pi b x}{l}\right)dx \tag{15}$$

$$= \frac{l}{2}\delta_{ab} \tag{16}$$

We therefore get

$$\int_0^l dx\, c^2\left(\frac{\partial u}{\partial x}\right)^2 = \sum_{k=1}^{\infty} c^2\frac{\omega_k^2}{c^2}\int_0^l q_k^2\cos^2\frac{\omega_k x}{c}dx \tag{17}$$

$$= \frac{l}{2}\sum_{k=1}^{\infty}\omega_k^2 q_k^2 \tag{18}$$

Putting the terms together we get

$$L = \frac{1}{2l}\sum_{k=1}^{\infty}p_k^2 - \frac{l}{2}\sum_{k=1}^{\infty}\omega_k^2 q_k^2 \tag{19}$$

The total Hamiltonian is given by

$$H = \sum_{k=1}^{\infty}p_k\dot{q}_k - L \tag{20}$$

$$= \sum_{k=1}^{\infty}\frac{p_k^2}{l} - \frac{1}{2l}\sum_{k=1}^{\infty}p_k^2 + \frac{l}{2}\sum_{k=1}^{\infty}\omega_k^2 q_k^2 \tag{21}$$

$$= \frac{1}{2l}\sum_{k=1}^{\infty}\left(p_k^2 + l^2\omega_k^2 q_k^2\right) \tag{22}$$

We now introduce the operators (essentially the same as the raising and lowering operators for the harmonic oscillator):

$$q_k = \sqrt{\frac{\hbar}{2l\omega_k}}\left(a^\dagger e^{i\omega_k t} + a_k e^{-i\omega_k t}\right) \tag{23}$$

$$p_k = l\dot{q}_k \tag{24}$$

$$= i\sqrt{\frac{l\hbar\omega_k}{2}}\left(a^\dagger e^{i\omega_k t} - a_k e^{-i\omega_k t}\right) \tag{25}$$

If we now interpret q_k and p_k as operators and require the usual commutation relation between position and momentum:

$$[q_k, p_j] = i\hbar\delta_{kj} \tag{26}$$

we can work out the commutators of a_k and $a_k^\dagger$. We *could* do this by inverting the above equations to express a_k and $a_k^\dagger$ in terms of q_k and p_k, but we can also just calculate $[q_k, p_j]$ in terms of the a_k and $a_k^\dagger$ operators. We have

$$[q_k, p_j] = \sqrt{\frac{\omega_j}{\omega_k}} \frac{i\hbar}{2} \left\{ \left[a_k, a_j^\dagger\right] e^{i(\omega_j - \omega_k)t} - [a_k, a_j]\, e^{-i(\omega_j + \omega_k)t} + \right. \tag{27}$$

$$\left. \left[a_k^\dagger, a_j^\dagger\right] e^{i(\omega_j + \omega_k)t} - \left[a_k^\dagger, a_j\right] e^{-i(\omega_j - \omega_k)t} \right\} \tag{28}$$

If we require this to be $i\hbar\delta_{kj}$, then because ω_k is always positive, the middle two terms must be zero, so we have

$$[a_k, a_j] = \left[a_k^\dagger, a_j^\dagger\right] = 0 \tag{29}$$

Also, if $j \neq k$, then the first and last terms will have an exponential term in them, so these terms too must be zero in this case. Finally, if $j = k$, the exponentials in the first and last terms are 1, so we get in this case

$$[q_k, p_k] = \frac{i\hbar}{2} \left(\left[a_k, a_k^\dagger\right] - \left[a_k^\dagger, a_k\right] \right) = i\hbar \tag{30}$$

which is true if

$$\left[a_k, a_k^\dagger\right] = 1 \tag{31}$$

Thus we get the final result

$$\left[a_k, a_j^\dagger\right] = \delta_{kj} \tag{32}$$

Finally, we can express the Hamiltonian in terms of a_k and $a_k^\dagger$ by just substituting 25 into 22. We get

$$H = \frac{1}{2l} \sum_{k=1}^{\infty} \left(-\frac{\hbar l \omega_k}{2} \left[a_k^{\dagger 2} e^{2i\omega_k t} + a_k^2 e^{-2i\omega_k t} - a_k^\dagger a_k - a_k a_k^\dagger \right] \right. \tag{33}$$

$$\left. + \frac{\hbar l \omega_k}{2} \left[a_k^{\dagger 2} e^{2i\omega_k t} + a_k^2 e^{-2i\omega_k t} + a_k^\dagger a_k + a_k a_k^\dagger \right] \right) \tag{34}$$

$$= \frac{\hbar}{2} \sum_{k=1}^{\infty} \omega_k \left(a_k^\dagger a_k + a_k a_k^\dagger \right) \tag{35}$$