

## STRESS-ENERGY TENSOR: 3 EXAMPLES

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Post date: 19 May 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.8.

In the derivation of Noether's theorem, L&P define the stress-energy tensor as

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \partial^\nu \Phi^A - g^{\mu\nu} \mathcal{L} \quad (1)$$

where there is a sum over the index  $A$ , which labels the independent fields  $\Phi^A$ .  $\mathcal{L}$  is the Lagrangian density and  $g^{\mu\nu}$  is the metric tensor in flat space-time. The stress-energy tensor is used to define a current

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \delta \Phi^A - T^{\mu\nu} \delta x_\nu \quad (2)$$

If the Lagrangian is invariant under some transformation (that is, it has a symmetry under this transformation), then Noether's theorem states that the corresponding current is conserved, which means that

$$\partial_\mu J^\mu = 0 \quad (3)$$

Here, we'll work out  $T^{\mu\nu}$  for three particular Lagrangians.

**Example 1.** First, we'll look at the real scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\alpha \phi) (\partial^\alpha \phi) - \frac{1}{2} m^2 \phi^2 - V(\phi) \quad (4)$$

$$= \frac{1}{2} (\partial_\alpha \phi) g^{\alpha\beta} (\partial_\beta \phi) - \frac{1}{2} m^2 \phi^2 - V(\phi) \quad (5)$$

We can drop the index  $A$  in the first term in 1 since there is only one field. Thus we have

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial^\nu \Phi = \left[ \frac{1}{2} \delta_{\alpha\mu} g^{\alpha\beta} (\partial_\beta \phi) + \frac{1}{2} g^{\alpha\beta} \delta_{\beta\mu} (\partial_\beta \phi) \right] \partial^\nu \phi \quad (6)$$

$$= \left[ \frac{1}{2} g^{\mu\beta} (\partial_\beta \phi) + \frac{1}{2} g^{\alpha\mu} (\partial_\beta \phi) \right] \partial^\nu \phi \quad (7)$$

$$= \partial^\mu \phi \partial^\nu \phi \quad (8)$$

The tensor is then

$$T^{\mu\nu} \equiv \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \quad (9)$$

**Example 2.** Now, we look at the complex scalar field with Lagrangian

$$\mathcal{L} = (\partial_\alpha \phi)^\dagger (\partial^\alpha \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi) \quad (10)$$

$$= (\partial_\alpha \phi)^\dagger g^{\alpha\beta} (\partial_\beta \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi) \quad (11)$$

We now have two fields ( $\phi$  and  $\phi^\dagger$ ) so the sum over  $A$  in 1 contains two terms. We have

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \partial^\nu \Phi^A = (\partial_\alpha \phi)^\dagger g^{\alpha\beta} \delta_{\beta\mu} \partial^\nu \phi + \delta_{\alpha\mu} g^{\alpha\beta} \partial_\beta \phi (\partial^\nu \phi)^\dagger \quad (12)$$

$$= (\partial^\mu \phi)^\dagger \partial^\nu \phi + \partial^\mu \phi (\partial^\nu \phi)^\dagger \quad (13)$$

which gives the tensor:

$$T^{\mu\nu} = (\partial^\mu \phi)^\dagger \partial^\nu \phi + \partial^\mu \phi (\partial^\nu \phi)^\dagger - g^{\mu\nu} \mathcal{L} \quad (14)$$

**Example 3.** Finally, we look at the electromagnetic Lagrangian with zero current ( $j^\mu = 0$ ):

$$\mathcal{L} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \quad (15)$$

with

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad (16)$$

As always, it's important to keep track of the positions of the various indexes. Since the derivative in 1 involves a derivative of a field with an upper index with respect to a coordinate with lower index, we convert the Lagrangian into this form.

$$\mathcal{L} = -\frac{1}{4} \left( g^{\alpha\gamma} \partial_\gamma A^\beta - g^{\beta\delta} \partial_\delta A^\alpha \right) \left( g_{\beta\epsilon} \partial_\alpha A^\epsilon - g_{\alpha\theta} \partial_\beta A^\theta \right) \quad (17)$$

Taking the derivative of  $\mathcal{L}$  requires the product rule. We have

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\rho)} = -\frac{1}{4} \left( g^{\alpha\gamma} \delta_{\gamma\mu} \delta_{\beta\rho} - g^{\beta\delta} \delta_{\delta\mu} \delta_{\alpha\rho} \right) \left( g_{\beta\varepsilon} \partial_\alpha A^\varepsilon - g_{\alpha\theta} \partial_\beta A^\theta \right) \quad (18)$$

$$-\frac{1}{4} \left( g^{\alpha\gamma} \partial_\gamma A^\beta - g^{\beta\delta} \partial_\delta A^\alpha \right) \left( g_{\beta\varepsilon} \delta_{\alpha\mu} \delta_{\varepsilon\rho} - g_{\alpha\theta} \delta_{\beta\mu} \delta_{\theta\rho} \right) \quad (19)$$

Using the Kronecker deltas to reduce the indexes (including the relation  $g^{\beta\delta} g_{\beta\varepsilon} = \delta_{\delta\varepsilon}$ ; I've used two lower indexes for Kronecker deltas since for these it doesn't matter where you place the indexes - the results are always the same) gives us

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\rho)} = -\frac{1}{4} \left( \partial^\mu A_\rho - \partial_\rho A^\mu - \partial_\rho A^\mu + \partial^\mu A_\rho \right) \quad (20)$$

$$-\frac{1}{4} \left( \partial^\mu A_\rho - \partial_\rho A^\mu - \partial_\rho A^\mu + \partial^\mu A_\rho \right) \quad (21)$$

$$= - \left( \partial^\mu A_\rho - \partial_\rho A^\mu \right) \quad (22)$$

The first term in 1 is then

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \partial^\nu \Phi^A = - \left( \partial^\mu A_\rho - \partial_\rho A^\mu \right) \partial^\nu A^\rho \quad (23)$$

$$= - \left( \partial^\mu A^\rho - \partial^\rho A^\mu \right) \partial^\nu A_\rho \quad (24)$$

$$= -F^{\mu\rho} \partial^\nu A_\rho \quad (25)$$

This gives the stress-energy tensor as

$$T^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L} \quad (26)$$

#### PINGBACKS

Pingback: Noether's theorem - internal symmetry of complex scalar field

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Pingback: Noether's theorem - traceless stress-energy tensor

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