STRESS-ENERGY TENSOR: 3 EXAMPLES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.8.

In the derivation of Noether's theorem, L&P define the stress-energy tensor as

$$T^{\mu\nu} \equiv \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \Phi^A\right)} \partial^\nu \Phi^A - g^{\mu\nu} \mathscr{L} \tag{1}$$

where there is a sum over the index A, which labels the independent fields Φ^A. $\mathscr{L}$ is the Lagrangian density and $g^{\mu\nu}$ is the metric tensor in flat spacetime. The stress-energy tensor is used to define a current

$$J^\mu \equiv \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \Phi^A\right)} \delta \Phi^A - T^{\mu\nu} \delta x_\nu \tag{2}$$

If the Lagrangian is invariant under some transformation (that is, it has a symmetry under this transformation), then Noether's theorem states that the corresponding current is conserved, which means that

$$\partial_\mu J^\mu = 0 \tag{3}$$

Here, we'll work out $T^{\mu\nu}$ for three particular Lagrangians.

Example 1. First, we'll look at the real scalar field with the Lagrangian

$$\mathscr{L} = \frac{1}{2} \left(\partial_\alpha \phi\right) \left(\partial^\alpha \phi\right) - \frac{1}{2} m^2 \phi^2 - V(\phi) \tag{4}$$

$$= \frac{1}{2} \left(\partial_\alpha \phi\right) g^{\alpha\beta} \left(\partial_\beta \phi\right) - \frac{1}{2} m^2 \phi^2 - V(\phi) \tag{5}$$

We can drop the index A in the first term in 1 since there is only one field. Thus we have

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}\partial^\nu \Phi = \left[\frac{1}{2}\delta_{\alpha\mu}g^{\alpha\beta}\left(\partial_\beta\phi\right)+\frac{1}{2}g^{\alpha\beta}\delta_{\beta\mu}\left(\partial_\beta\phi\right)\right]\partial^\nu\phi \tag{6}$$

$$= \left[\frac{1}{2}g^{\mu\beta}\left(\partial_\beta\phi\right)+\frac{1}{2}g^{\alpha\mu}\left(\partial_\beta\phi\right)\right]\partial^\nu\phi \tag{7}$$

$$= \partial^\mu\phi\partial^\nu\phi \tag{8}$$

The tensor is then

$$T^{\mu\nu} \equiv \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathscr{L} \tag{9}$$

Example 2. Now, we look at the complex scalar field with Lagrangian

$$\mathscr{L} = \left(\partial_\alpha\phi\right)^\dagger\left(\partial^\alpha\phi\right) - m^2\phi^\dagger\phi - V\left(\phi^\dagger\phi\right) \tag{10}$$

$$= \left(\partial_\alpha\phi\right)^\dagger g^{\alpha\beta}\left(\partial_\beta\phi\right) - m^2\phi^\dagger\phi - V\left(\phi^\dagger\phi\right) \tag{11}$$

We now have two fields (ϕ and $\phi^\dagger$) so the sum over A in 1 contains two terms. We have

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)}\partial^\nu \Phi^A = \left(\partial_\alpha\phi\right)^\dagger g^{\alpha\beta}\delta_{\beta\mu}\partial^\nu\phi + \delta_{\alpha\mu}g^{\alpha\beta}\partial_\beta\phi\left(\partial^\nu\phi\right)^\dagger \tag{12}$$

$$= \left(\partial^\mu\phi\right)^\dagger\partial^\nu\phi + \partial^\mu\phi\left(\partial^\nu\phi\right)^\dagger \tag{13}$$

which gives the tensor:

$$T^{\mu\nu} = \left(\partial^\mu\phi\right)^\dagger\partial^\nu\phi + \partial^\mu\phi\left(\partial^\nu\phi\right)^\dagger - g^{\mu\nu}\mathscr{L} \tag{14}$$

Example 3. Finally, we look at the electromagnetic Lagrangian with zero current ($j^\mu = 0$):

$$\mathscr{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \tag{15}$$

with

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{16}$$

As always, it's important to keep track of the positions of the various indexes. Since the derivative in 1 involves a derivative of a field with an upper index with respect to a coordinate with lower index, we convert the Lagrangian into this form.

$$\mathscr{L} = -\frac{1}{4}\left(g^{\alpha\gamma}\partial_\gamma A^\beta - g^{\beta\delta}\partial_\delta A^\alpha\right)\left(g_{\beta\epsilon}\partial_\alpha A^\epsilon - g_{\alpha\theta}\partial_\beta A^\theta\right) \tag{17}$$

Taking the derivative of $\mathscr{L}$ requires the product rule. We have

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\Phi^{\rho}\right)}=-\frac{1}{4}\left(g^{\alpha\gamma}\delta_{\gamma\mu}\delta_{\beta\rho}-g^{\beta\delta}\delta_{\delta\mu}\delta_{\alpha\rho}\right)\left(g_{\beta\epsilon}\partial_{\alpha}A^{\epsilon}-g_{\alpha\theta}\partial_{\beta}A^{\theta}\right) \tag{18}$$

$$-\frac{1}{4}\left(g^{\alpha\gamma}\partial_{\gamma}A^{\beta}-g^{\beta\delta}\partial_{\delta}A^{\alpha}\right)\left(g_{\beta\epsilon}\delta_{\alpha\mu}\delta_{\epsilon\rho}-g_{\alpha\theta}\delta_{\beta\mu}\delta_{\theta\rho}\right) \tag{19}$$

Using the Kronecker deltas to reduce the indexes (including the relation $g^{\beta\delta}g_{\beta\epsilon}=\delta_{\delta\epsilon}$; I've used two lower indexes for Kronecker deltas since for these it doesn't matter where you place the indexes - the results are always the same) gives us

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\Phi^{\rho}\right)}=-\frac{1}{4}\left(\partial^{\mu}A_{\rho}-\partial_{\rho}A^{\mu}-\partial_{\rho}A^{\mu}+\partial^{\mu}A_{\rho}\right) \tag{20}$$

$$-\frac{1}{4}\left(\partial^{\mu}A_{\rho}-\partial_{\rho}A^{\mu}-\partial_{\rho}A^{\mu}+\partial^{\mu}A_{\rho}\right) \tag{21}$$

$$=-\left(\partial^{\mu}A_{\rho}-\partial_{\rho}A^{\mu}\right) \tag{22}$$

The first term in 1 is then

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\Phi^{A}\right)}\partial^{\nu}\Phi^{A}=-\left(\partial^{\mu}A_{\rho}-\partial_{\rho}A^{\mu}\right)\partial^{\nu}A^{\rho} \tag{23}$$

$$=-\left(\partial^{\mu}A^{\rho}-\partial^{\rho}A^{\mu}\right)\partial^{\nu}A_{\rho} \tag{24}$$

$$=-F^{\mu\rho}\partial^{\nu}A_{\rho} \tag{25}$$

This gives the stress-energy tensor as

$$T^{\mu\nu}=-F^{\mu\rho}\partial^{\nu}A_{\rho}-g^{\mu\nu}\mathscr{L} \tag{26}$$

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