NOETHER'S THEOREM - INTERNAL SYMMETRY OF COMPLEX SCALAR FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.9.

An internal symmetry is one in which the Lagrangian (or action) is invariant under the change of its fields at the same space-time point. That is, the symmetry does not depend on a change of the coordinate system. The field transformation is then given by

There is no implied sum over r.

$$\Phi^A(x) \to \Phi^A(x) + f_r^A(x)\,\delta\varepsilon_r \tag{1}$$

Here, $\delta\varepsilon_r$ are infinitesimal quantities that don't depend on space-time, with the index r labelling the symmetry. A given system can have several such symmetries, each of which results in a transformation of the fields as given by 1. The quantity $f_r^A(x)$ is some function of the fields and their derivatives (and through them, a function of space-time as well). The index A labels the particular field.

If the Lagrangian is invariant under such a transformation, L&P state that the corresponding conserved current, according to Noether's theorem, is given by

$$J_r^\mu = \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\frac{\delta\Phi^A}{\delta\varepsilon_r} \tag{2}$$

with an implied sum over the field index A. I'm not clear on where this comes from, as it is just stated as equation 2.64 in the text.

In any case, if we accept this definition of the current, we can do a couple of examples using it.

Consider again the complex scalar field we looked at earlier. The Lagrangian is

$$\mathscr{L} = \left(\partial_\mu\phi\right)^\dagger\left(\partial^\mu\phi\right) - m^2\phi^\dagger\phi - V\left(\phi^\dagger\phi\right) \tag{3}$$

Since the fields always appear in products of the field with its complex conjugate, the Lagrangian is invariant under the transformations

$$\phi \to e^{-iq\theta}\phi \tag{4}$$
$$\phi^\dagger \to e^{iq\theta}\phi^\dagger \tag{5}$$

where q is a parameter (not necessarily infinitesimal) and θ is an infinitesimal parameter (so it corresponds to $\delta\varepsilon$ in 1). In the infinitesimal case, we can expand the exponentials to first order in θ:

$$e^{-iq\theta} = 1 - iq\theta + \mathcal{O}\left(\theta^2\right) \tag{6}$$
$$e^{iq\theta} = 1 + iq\theta + \mathcal{O}\left(\theta^2\right) \tag{7}$$

Thus the changes in the fields are

$$\delta\phi = -iq\theta\phi \tag{8}$$
$$\delta\phi^\dagger = iq\theta\phi^\dagger \tag{9}$$

Applying 2 to 3 we have

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)} = \left(\partial^\mu\phi\right)^\dagger \tag{10}$$
$$\frac{\delta\phi}{\theta} = -iq\phi \tag{11}$$
$$\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^\dagger\right)} = \left(\partial^\mu\phi\right) \tag{12}$$
$$\frac{\delta\phi^\dagger}{\theta} = iq\phi^\dagger \tag{13}$$

Putting it all together, we get

$$j^\mu = iq\left(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger\right) \tag{14}$$

Since this is supposed to be a conserved current, it should follow that

$$\partial_\mu j^\mu = 0 \tag{15}$$

We can verify this as follows. Taking the derivative of 14 we have

$$\partial_\mu j^\mu = iq\left(\partial_\mu\phi^\dagger\partial^\mu\phi + \phi^\dagger\partial_\mu\partial^\mu\phi - \partial_\mu\phi\partial^\mu\phi^\dagger - \phi\partial_\mu\partial^\mu\phi^\dagger\right) \tag{16}$$

$$= iq\left(\phi^\dagger\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^\dagger\right) \tag{17}$$

$$= iq\left(\phi^\dagger\Box\phi - \phi\Box\phi^\dagger\right) \tag{18}$$

We can relate the last line to the Euler-Lagrange equations for a complex field

$$\left(\Box + m^2\right)\phi^\dagger = -\frac{\partial V}{\partial\phi} \tag{19}$$

The potential V is a function of the product $\phi^\dagger\phi$ so using the chain rule, we have

$$\frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial\left(\phi^\dagger\phi\right)}\frac{\partial\left(\phi^\dagger\phi\right)}{\partial\phi} = \frac{\partial V}{\partial\left(\phi^\dagger\phi\right)}\phi^\dagger \tag{20}$$

Therefore

$$\Box\phi^\dagger = -m^2\phi^\dagger - \frac{\partial V}{\partial\left(\phi^\dagger\phi\right)}\phi^\dagger \tag{21}$$

$$\Box\phi = -m^2\phi - \frac{\partial V}{\partial\left(\phi^\dagger\phi\right)}\phi \tag{22}$$

Combining these, we get

$$\phi^\dagger\Box\phi - \phi\Box\phi^\dagger = -m^2\phi\phi^\dagger - \frac{\partial V}{\partial\left(\phi^\dagger\phi\right)}\phi\phi^\dagger + m^2\phi^\dagger\phi + \frac{\partial V}{\partial\left(\phi^\dagger\phi\right)}\phi^\dagger\phi \tag{23}$$

$$= 0 \tag{24}$$

Thus the current j^μ satisfies 15.

We can now introduce the electromagnetic field into the Lagrangian. L&P simply state the Lagrangian for this case as

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[\left(\partial^\mu - iqA^\mu\right)\phi^\dagger\right]\left[\left(\partial_\mu + iqA_\mu\right)\phi\right] \tag{25}$$

$$-m^2\phi^\dagger\phi - V\left(\phi^\dagger\phi\right) \tag{26}$$

Although no derivation of $\mathscr{L}$ is given, it appears to be related to the classical electromagnetic Lagrangian we met earlier. Essentially, the momentum p^μ is replaced by $p^\mu - qA^\mu$, and since the momentum operator in quantum mechanics is given by $p^\mu = -i\partial^\mu$, this results in the second term in 25.

This isn't an exact derivation, but at least it makes the Lagrangian appear reasonable.

Under the same transformation as in 4, this new Lagrangian is again invariant, so we can calculate the Noether current. If we apply 2 we can note first that we're assuming there is no variation in the electromagnetic fields, so $\delta A^{\mu}=0$, which means we can ignore any derivatives with respect to A^{μ} or its derivatives. If we write out the relevant terms from 25 (that is, only those terms that involve derivatives of ϕ or $\phi^{\dagger}$), we have

$$\mathscr{L}\to\left(\partial_{\mu}\phi\right)^{\dagger}\left(\partial^{\mu}\phi\right)+iqA_{\mu}\phi\partial^{\mu}\phi^{\dagger}-iqA^{\mu}\phi^{\dagger}\partial_{\mu}\phi \tag{27}$$

We now have

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\phi\right)}=\left(\partial^{\mu}\phi\right)^{\dagger}-iqA^{\mu}\phi^{\dagger} \tag{28}$$

$$\frac{\delta\phi}{\theta}=-iq\phi \tag{29}$$

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\phi^{\dagger}\right)}=\left(\partial^{\mu}\phi\right)+iqA^{\mu}\phi \tag{30}$$

$$\frac{\delta\phi^{\dagger}}{\theta}=iq\phi^{\dagger} \tag{31}$$

Combining everything gives

$$j^{\mu}=iq\left(\phi^{\dagger}\partial^{\mu}\phi-\phi\partial^{\mu}\phi^{\dagger}\right)+(iq)^{2}A^{\mu}\phi\phi^{\dagger}+(iq)^{2}A^{\mu}\phi\phi^{\dagger} \tag{32}$$

$$=iq\left(\phi^{\dagger}\partial^{\mu}\phi-\phi\partial^{\mu}\phi^{\dagger}+2iqA^{\mu}\phi\phi^{\dagger}\right) \tag{33}$$

We've seen earlier that applying the Euler-Lagrange equations to a pure electromagnetic Lagrangian results in Maxwell's equations. We can also apply the Euler-Lagrange equations to the combined Lagrangian 25. To do this, we calculate the Euler-Lagrange equations for the electromagnetic field components A^{μ}. The equations are

$$\partial_{\rho}\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_{\rho}A_{\mu}\right)}\right)=\frac{\partial\mathscr{L}}{\partial A_{\mu}} \tag{34}$$

To work out the LHS, note that the only term in 25 that involves $\partial_{\rho}A^{\mu}$ is the first term, namely $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. We've already worked out the required derivative when calculating the stress-energy tensor so we have

$$\frac{\partial\mathscr{L}}{\partial\left(\partial_\rho A_\mu\right)}=-F^{\rho\mu} \tag{35}$$

To work out the RHS of 34, we isolate those terms in 25 that involve A_μ:

$$\mathscr{L}\to -iqA^\mu\phi^\dagger\partial_\mu\phi+iqA_\mu\phi\partial^\mu\phi^\dagger+q^2\phi^\dagger\phi A^\mu A_\mu \tag{36}$$

Taking the derivative on the RHS of 34 we have

$$\frac{\partial\mathscr{L}}{\partial A_\mu}=-iq\phi^\dagger\partial^\mu\phi+iq\phi\partial^\mu\phi^\dagger+2q^2\phi^\dagger\phi A^\mu \tag{37}$$

$$=-iq\left(\phi^\dagger\partial^\mu\phi-\phi\partial^\mu\phi^\dagger+2iqA^\mu\phi\phi^\dagger\right) \tag{38}$$

Combining the results we have

$$-\partial_\rho F^{\rho\mu}=-iq\left(\phi^\dagger\partial^\mu\phi-\phi\partial^\mu\phi^\dagger+2iqA^\mu\phi\phi^\dagger\right) \tag{39}$$

$$\partial_\mu F^{\mu\nu}=iq\left(\phi^\dagger\partial^\nu\phi-\phi\partial^\nu\phi^\dagger+2iqA^\nu\phi\phi^\dagger\right) \tag{40}$$

$$=j^\nu \tag{41}$$

where we've relabelled the indexes in the second line, and then compared the result with 33. Thus the current in this case is the same as the conserved current worked out using Noether's theorem above.

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