NOETHER'S THEOREM - INTERNAL SYMMETRY AND SCALED SPACETIME

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.10.

As another example of an internal symmetry, but this time combined with a change in the coordinates, we'll consider the real scalar field with $m=0$ and a potential given by $V=\lambda\phi^4$. The Lagrangian is

$$\mathscr{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\lambda\phi^4 \tag{1}$$

If we now impose the transformations

$$x^\mu \to bx^\mu \tag{2}$$

$$\phi \to \frac{\phi}{b} \tag{3}$$

where b is a constant, then the Lagrangian becomes

$$\mathscr{L}\to\frac{1}{b^4}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\lambda\phi^4\right) \tag{4}$$

The first term follows from

$$\partial_\mu=\frac{\partial}{\partial x^\mu}\to\frac{1}{b}\frac{\partial}{\partial x^\mu} \tag{5}$$

Although the Lagrangian isn't invariant under this transformation, the action is, as it is defined as

$$\mathscr{A}=\int d^4x\,\mathscr{L} \tag{6}$$

and the integration increment transforms as

$$d^4x\to b^4d^4x \tag{7}$$

so that the product $d^4x\,\mathscr{L}$ remains invariant.

To apply Noether's theorem to this symmetry, we note first that the formulas given in the textbook apply to an infinitesimal transformation, while the constant b above is not an infinitesimal. However, we can consider the transformation where $b=1-\varepsilon$, with ε an infinitesimal quantity. In this case, the transformation *is* infinitesimal so we can apply Noether's theorem. In that case, to first order in ε, we have

$$x^\mu \to x^\mu - \varepsilon x^\mu \tag{8}$$

$$\phi \to \phi + \varepsilon\phi \tag{9}$$

so we have

$$\delta x^\mu = -\varepsilon x^\mu \tag{10}$$

$$\delta\phi = \varepsilon\phi \tag{11}$$

The transformation in this case is an internal symmetry (since the variation occurs at a single spacetime point) but now $\delta x^\mu \neq 0$. We can modify L&P's equation 2.64 by dividing the original definition of the conserved current, which is

$$J^\mu \equiv \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\delta\Phi^A - T^{\mu\nu}\delta x_\nu \tag{12}$$

The recipe for dealing with an internal symmetry is to divide this current by the infinitesimal parameter, which in this case is ε, so we get

$$j^\mu = \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\frac{\delta\phi}{\varepsilon} - T^{\mu\nu}\frac{\delta x_\nu}{\varepsilon} \tag{13}$$

$$= \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\phi + T^{\mu\nu}x_\nu \tag{14}$$

The stress-energy tensor is given by

$$T^{\mu\nu} \equiv \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\partial^\nu\Phi^A - g^{\mu\nu}\mathscr{L} \tag{15}$$

which becomes, in this case

$$T^{\mu\nu} \equiv \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\partial^\nu\phi - g^{\mu\nu}\mathscr{L} \tag{16}$$

$$= \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathscr{L} \tag{17}$$

Inserting this into 14 we have

$$
\begin{aligned}
j^{\mu} &= \phi \partial^{\mu} \phi + x_{\nu} \partial^{\mu} \phi \partial^{\nu} \phi - x_{\nu} g^{\mu\nu} \mathscr{L} && (18) \\
&= \phi \partial^{\mu} \phi + x_{\nu} \partial^{\mu} \phi \partial^{\nu} \phi - x^{\mu} \mathscr{L} && (19) \\
&= \partial^{\nu} \left(x_{\nu} \phi \right) \partial^{\mu} \phi - x^{\mu} \mathscr{L} && (20)
\end{aligned}
$$

where the last line condenses the second line by using the product rule.