NOETHER'S THEOREM - TRACELESS STRESS-ENERGY TENSOR

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 2, Problem 2.11.

Here's another example of the application of Noether's theorem. This time we consider a transformation in which the action (but not necessarily the Lagrangian) is invariant under spacetime translation and spacetime dilation. That is, the translation is given by

$$x^{\mu} \to x^{\mu} + a^{\mu} \tag{1}$$

where the a^{μ} are constants, and the dilation is given by

$$x^{\mu} \to b x^{\mu} \tag{2}$$

where b is another constant. The field in this case is assumed not to change, that is

$$\Phi \to \Phi \tag{3}$$

For infinitesimal transformations, L&P show in their equation 2.43 that the variation in the action is given by

$$\delta \mathscr{A} = \int_{\Omega} d^4x \, \partial_{\mu} \left[\frac{\partial \mathscr{L}}{\partial \left(\partial_{\mu} \Phi^A \right)} \delta \Phi^A - T^{\mu\nu} \delta x_{\nu} \right] \tag{4}$$

where the index A labels the field (we have only one field here) and $T^{\mu\nu}$ is the stress-energy tensor. In this problem, we won't need the explicit form of $T^{\mu\nu}$. The integral is performed over the spacetime volume Ω which should contain the entire region pertaining to the problem.

With the conditions above, we have $\delta\Phi = 0$, so the first term in the integrand of 4 is zero. To get a condition on the second term, we need δx_{ν}, which arises from the two transformations above. Since we must deal with infinitesimal translations, we assume that a^{μ} is infinitesimal, and for b we have

$$b = 1 - \varepsilon \tag{5}$$

where ε is infinitesimal. In that case, the total infinitesimal increment is

$$\delta x_\nu = a_\nu - \varepsilon x_\nu \tag{6}$$

The variation in the action is then

$$\delta\mathscr{A} = -\int_\Omega d^4x\,\partial_\mu\left(T^{\mu\nu}\delta x_\nu\right) \tag{7}$$

$$= -\int_\Omega d^4x\,\partial_\mu\left(T^{\mu\nu}\left(a_\nu - \varepsilon x_\nu\right)\right) \tag{8}$$

$$= -\int_\Omega d^4x\,\partial_\mu\left(T^{\mu\nu}\left(a_\nu - \varepsilon g_{\nu\rho}x^\rho\right)\right) \tag{9}$$

Since a_ν and ε are constants, we can evaluate the derivative to get

$$\delta\mathscr{A} = -\int_\Omega d^4x\left[\partial_\mu T^{\mu\nu}\left(a_\nu - \varepsilon g_{\nu\rho}x^\rho\right) - \varepsilon g_{\nu\rho}T^{\mu\nu}\delta^\rho_\mu\right] \tag{10}$$

$$= -\int_\Omega d^4x\left[\partial_\mu T^{\mu\nu}\left(a_\nu - \varepsilon g_{\nu\rho}x^\rho\right) - \varepsilon g_{\nu\mu}T^{\mu\nu}\right] \tag{11}$$

$$= -\int_\Omega d^4x\left[\partial_\mu T^{\mu\nu}\left(a_\nu - \varepsilon g_{\nu\rho}x^\rho\right) - \varepsilon T^\mu{}_\mu\right] \tag{12}$$

We now require the action to be invariant, so that $\delta\mathscr{A} = 0$. Since a_ν is an arbitrary infinitesimal displacement, we must have

$$\partial_\mu T^{\mu\nu} = 0 \tag{13}$$

This is the same result as in L&P's equation 2.54, which was derived for the case of translation alone.

to eliminate the first term in 12. The other parameter ε is also arbitrary so anything multiplying it must also be zero. The second term in 12 thus gives us the condition

$$T^\mu{}_\mu = 0 \tag{14}$$

$T^\mu{}_\mu$ is the sum of the diagonal elements in the stress-energy tensor.

That is, the stress-energy tensor is traceless in this case. Notice that this result doesn't depend on the form of the Lagrangian or the stress-energy tensor; it is derived purely from the requirement that the action be invariant under the given transformations.