NATURAL UNITS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapters 1 & 3, Problems 3.1 - 3.2.

In quantum field theory, it is usual to define the unit system so that Planck's constant and the speed of light are both equal to 1, and both dimensionless. That is, we define

$$\hbar = c = 1 \tag{1}$$

This system of units is known as *natural units*. The inclusion of $\hbar = 1$ goes one step further than with relativistic units which we used in general relativity, where only $c = 1$ since general relativity doesn't use $\hbar$ at all.

To convert expressions in natural units back to conventional (that is, SI or CGS) units, we need to insert powers of $\hbar$ and/or c until the quantity's conventional units are restored. The required conversions are given in L&P's table 1.1 on page 10. In natural units, there is only one actual unit, which is usually taken to be mass. Every other unit can be expressed as mass to some power. To see how this comes about, we start with 1. Since $c = 1$ and is dimensionless, we see that length and time must have the same units (since c is a velocity, which is $[\text{length}]\left[\text{time}^{-1}\right]$). We'll write this as

$$L = T \tag{2}$$

The other condition, $\hbar = 1$, sorts out the other units. Planck's constant has the units of angular momentum, or action, which is $[\text{energy}] \times [\text{time}]$. Energy is mass times velocity squared (as in the kinetic energy), so we have

$$[\hbar] = [\text{mass}]\left[\text{length}^2\right]\left[\text{time}^{-1}\right] \tag{3}$$

Since we know that length and time have the same units, we can cancel one of the length factors with the time factor, so

$$[\hbar] = [\text{mass}][\text{length}] \tag{4}$$

Since this must be dimensionless, we see that

$$[\text{length}] = \left[\text{mass}^{-1}\right] \tag{5}$$

Therefore, taking mass as the fundamental unit, we have

$$[\text{energy}] = [\text{mass}] \tag{6}$$

$$[\text{length}] = \left[\text{mass}^{-1}\right] \tag{7}$$

$$[\text{time}] = \left[\text{mass}^{-1}\right] \tag{8}$$

$$[\text{velocity}] = 1 \tag{9}$$

$$[\text{linear momentum}] = [\text{mass}] \tag{10}$$

$$[\text{angular momentum}] = 1 \tag{11}$$

$$[\text{action}] = 1 \tag{12}$$

Here '1' indicates that the quantity is dimensionless.

As an example, consider the Lagrangian of a real scalar field with zero potential:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2 \tag{13}$$

We can work out the units of $\mathscr{L}$ by considering its role in the action, which in 4-dimensional space-time is

$$\mathscr{A} = \int d^4 x \; \mathscr{L} \tag{14}$$

Since the action is dimensionless and each dimension in spacetime has the units of $\left[\text{mass}^{-1}\right]$, we see that the dimensions of $\mathscr{L}$ must be

$$[\mathscr{L}] = \left[\text{mass}^4\right] \tag{15}$$

In a more general theory where space-time has N dimensions, the integration is over $d^N x$ so in this case

$$[\mathscr{L}] = \left[\text{mass}^N\right] \tag{16}$$

Using this in 13, we see that the field ϕ must then have units so that

$$\left[m^2 \phi^2\right] = \left[\text{mass}^N\right] \tag{17}$$

Since m has the units of mass, m^2 has the units of $\left[\text{mass}^2\right]$, so

$$\left[\phi^2\right] = \left[\text{mass}^{N-2}\right] \tag{18}$$

or

$$[\phi] = \left[\text{mass}^{(N-2)/2}\right] \tag{19}$$

The Hamiltonian for a field is given by

$$\mathscr{H} = \Pi\dot{\phi} - \mathscr{L} \tag{20}$$

where the conjugate momentum is given by

$$\Pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \tag{21}$$

From 13, this is

$$\Pi = \partial_0 \phi = \dot{\phi} \tag{22}$$

so the Hamiltonian is

$$\begin{aligned}
\mathscr{H} &= \Pi^2 - \mathscr{L} &(23)\\
&= \Pi^2 - \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 &(24)\\
&= \Pi^2 + \frac{1}{2}\left(-\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) &(25)\\
&= \frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) &(26)
\end{aligned}$$

The Hamiltonian has the same units as the Lagrangian, that is, $\left[\text{mass}^N\right]$.

The Hamiltonian $\mathscr{H}$ and Lagrangian $\mathscr{L}$ used here are both densities, so have units of energy per unit volume, where the volume is the spatial volume (excluding time) with $N-1$ dimensions. Since energy has the units of mass and inverse spatial volume has units of mass^{N-1}, this gives rise to the overall units being mass^N.

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