REAL SCALAR FIELD - FOURIER DECOMPOSITION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.3.

The real scalar field ϕ satisfies the Klein-Gordon equation

$$\left(\Box+m^2\right)\phi(x)=0 \tag{1}$$

where x represents the four-vector of space-time. The four-momentum p satisfies the condition

$$p^2=m^2 \tag{2}$$

so we can write the field as a Fourier integral over the momentum, subject to this condition. That is

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int d^4p\,\delta\left(p^2-m^2\right)A(p)\,e^{-ip\cdot x} \tag{3}$$

where $A(p)$ is a coefficient that weights the contribution from momentum p.

In L&P's section 3.3, they show how this integral may be converted to an integral over the spatial components p_i by using the delta function to do the integral over p_0. The result is

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2E_p}}\left(a(p)\,e^{-ip\cdot x}+a^{\dagger}(p)\,e^{ip\cdot x}\right) \tag{4}$$

where

$$E_p=p_0=+\sqrt{\mathbf{p}^2+m^2} \tag{5}$$

$$a(p)=\frac{A(p)}{\sqrt{2E_p}} \tag{6}$$

The conjugate momentum for this field is

$$\Pi(x) = \dot{\phi}(x) \tag{7}$$
$$= \frac{i}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{E_p}{2}} \left(-a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right) \tag{8}$$

Quantization is performed by interpreting ϕ, Π, a and $a^\dagger$ as operators. As operators, we need to know their commutators, which are derived by the usual prescription of converting classical Poisson brackets to commutators. In classical field theory, the Poisson bracket of a field and its conjugate momentum is given by

$$[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})]_P = \delta^3(\mathbf{x}-\mathbf{y}) \tag{9}$$
$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})]_P = 0 \tag{10}$$
$$[\Pi(t,\mathbf{x}), \Pi(t,\mathbf{y})]_P = 0 \tag{11}$$

[Recall that Lahiri & Pal's notation for a Poisson bracket is $[\phi,\Pi]_P$ rather than the more usual $\{\phi,\Pi\}$.] The recipe for converting a Poisson bracket to a commutator is to insert an i in front of the commutator (actually, an $i\hbar$ but we're using natural units with $\hbar = c = 1$). We therefore have

$$[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}) \tag{12}$$
$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = 0 \tag{13}$$
$$[\Pi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = 0 \tag{14}$$

By applying this commutator to 4 and 8, we can derive the commutators for a and $a^\dagger$, which turn out to be

$$\left[a(p), a^\dagger(p')\right] = \delta^3(\mathbf{p}-\mathbf{p}') \tag{15}$$
$$\left[a(p), a(p')\right] = 0 \tag{16}$$
$$\left[a^\dagger(p), a^\dagger(p')\right] = 0 \tag{17}$$

Using these, we can work out the commutator $[\phi(t,\mathbf{x}), \partial_i\phi(t,\mathbf{y})]$ as follows. We use 4 for $\phi(x)$ and its derivative:

$$\partial_i\phi(y) = \frac{i}{(2\pi)^{3/2}} \int \frac{d^3p'}{\sqrt{2E_{p'}}} p'_i \left(-a(p')e^{-ip'\cdot y} + a^\dagger(p')e^{ip'\cdot y}\right) \tag{18}$$

In calculating the commutator of 4 with 18 we see from the commutators 16 that only those terms involving the commutator of a with $a^{\dagger}$ will be non-zero, so we can ignore the other commutators. We get

$$\left[\phi\left(t,\mathbf{x}\right),\partial_{i}\phi\left(t,\mathbf{y}\right)\right]=\frac{i}{2\left(2\pi\right)^{3}}\int d^{3}p\int d^{3}p^{\prime}\left\{e^{-i\left(p\cdot x-p^{\prime}\cdot y\right)}p_{i}^{\prime}\left[a\left(p\right),a^{\dagger}\left(p^{\prime}\right)\right]-\right. \tag{19}$$

$$\left.e^{i\left(p\cdot x-p^{\prime}\cdot y\right)}p_{i}^{\prime}\left[a^{\dagger}\left(p\right),a\left(p^{\prime}\right)\right]\right\} \tag{20}$$

$$=\frac{i}{2\left(2\pi\right)^{3}}\int d^{3}p\int d^{3}p^{\prime}\left\{e^{-i\left(p\cdot x-p^{\prime}\cdot y\right)}p_{i}^{\prime}\left[a\left(p\right),a^{\dagger}\left(p^{\prime}\right)\right]+\right. \tag{21}$$

$$\left.e^{i\left(p\cdot x-p^{\prime}\cdot y\right)}p_{i}^{\prime}\left[a\left(p\right),a^{\dagger}\left(p^{\prime}\right)\right]\right\} \tag{22}$$

$$=\frac{i}{2\left(2\pi\right)^{3}}\int d^{3}p\int d^{3}p^{\prime}\left\{e^{-i\left(p\cdot x-p^{\prime}\cdot y\right)}p_{i}^{\prime}\delta^{3}\left(\mathbf{p}-\mathbf{p}^{\prime}\right)+\right. \tag{23}$$

$$\left.e^{i\left(p\cdot x-p^{\prime}\cdot y\right)}p_{i}^{\prime}\delta^{3}\left(\mathbf{p}-\mathbf{p}^{\prime}\right)\right\} \tag{24}$$

$$=\frac{i}{2\left(2\pi\right)^{3}}\int d^{3}p\,p_{i}\left(e^{-ip\cdot(x-y)}+e^{ip\cdot(x-y)}\right) \tag{25}$$

We now note that the integrand in the last line is an odd function of p. That is, if we replace p by $-p$ the integrand changes sign. The integral of an odd function over all p space is zero, so we have

$$\left[\phi\left(t,\mathbf{x}\right),\partial_{i}\phi\left(t,\mathbf{y}\right)\right]=0 \tag{26}$$

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