

REAL SCALAR FIELD - FOURIER DECOMPOSITION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.3.

The real scalar field ϕ satisfies the Klein-Gordon equation

$$(\square + m^2) \phi(x) = 0 \quad (1)$$

where x represents the four-vector of space-time. The four-momentum p satisfies the condition

$$p^2 = m^2 \quad (2)$$

so we can write the field as a Fourier integral over the momentum, subject to this condition. That is

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p \delta(p^2 - m^2) A(p) e^{-ip \cdot x} \quad (3)$$

where $A(p)$ is a coefficient that weights the contribution from momentum p .

In L&P's section 3.3, they show how this integral may be converted to an integral over the spatial components p_i by using the delta function to do the integral over p_0 . The result is

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \left(a(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x} \right) \quad (4)$$

where

$$E_p = p_0 = +\sqrt{\mathbf{p}^2 + m^2} \quad (5)$$

$$a(p) = \frac{A(p)}{\sqrt{2E_p}} \quad (6)$$

The conjugate momentum for this field is

$$\Pi(x) = \dot{\phi}(x) \quad (7)$$

$$= \frac{i}{(2\pi)^{3/2}} \int d^3 p \sqrt{\frac{E_p}{2}} \left(-a(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x} \right) \quad (8)$$

Quantization is performed by interpreting ϕ , Π , a and a^\dagger as operators. As operators, we need to know their commutators, which are derived by the usual prescription of converting classical Poisson brackets to commutators. In classical field theory, the Poisson bracket of a field and its conjugate momentum is given by

$$[\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})]_P = \delta^3(\mathbf{x} - \mathbf{y}) \quad (9)$$

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})]_P = 0 \quad (10)$$

$$[\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})]_P = 0 \quad (11)$$

[Recall that Lahiri & Pal's notation for a Poisson bracket is $[\phi, \Pi]_P$ rather than the more usual $\{\phi, \Pi\}$.] The recipe for converting a Poisson bracket to a commutator is to insert an i in front of the commutator (actually, an $i\hbar$ but we're using natural units with $\hbar = c = 1$). We therefore have

$$[\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}) \quad (12)$$

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = 0 \quad (13)$$

$$[\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = 0 \quad (14)$$

By applying this commutator to 4 and 8, we can derive the commutators for a and a^\dagger , which turn out to be

$$[a(p), a^\dagger(p')] = \delta^3(\mathbf{p} - \mathbf{p}') \quad (15)$$

$$[a(p), a(p')] = 0 \quad (16)$$

$$[a^\dagger(p), a^\dagger(p')] = 0 \quad (17)$$

Using these, we can work out the commutator $[\phi(t, \mathbf{x}), \partial_i \phi(t, \mathbf{y})]$ as follows. We use 4 for $\phi(x)$ and its derivative:

$$\partial_i \phi(y) = \frac{i}{(2\pi)^{3/2}} \int \frac{d^3 p'}{\sqrt{2E_{p'}}} p'_i \left(-a(p') e^{-ip' \cdot y} + a^\dagger(p') e^{ip' \cdot y} \right) \quad (18)$$

In calculating the commutator of 4 with 18 we see from the commutators 16 that only those terms involving the commutator of a with a^\dagger will be non-zero, so we can ignore the other commutators. We get

$$[\phi(t, \mathbf{x}), \partial_i \phi(t, \mathbf{y})] = \frac{i}{2(2\pi)^3} \int d^3 p \int d^3 p' \left\{ e^{-i(p \cdot x - p' \cdot y)} p'_i [a(p), a^\dagger(p')] - \right. \quad (19)$$

$$\left. e^{i(p \cdot x - p' \cdot y)} p'_i [a^\dagger(p), a(p')] \right\} \quad (20)$$

$$= \frac{i}{2(2\pi)^3} \int d^3 p \int d^3 p' \left\{ e^{-i(p \cdot x - p' \cdot y)} p'_i [a(p), a^\dagger(p')] + \right. \quad (21)$$

$$\left. e^{i(p \cdot x - p' \cdot y)} p'_i [a(p), a^\dagger(p')] \right\} \quad (22)$$

$$= \frac{i}{2(2\pi)^3} \int d^3 p \int d^3 p' \left\{ e^{-i(p \cdot x - p' \cdot y)} p'_i \delta^3(\mathbf{p} - \mathbf{p}') + \right. \quad (23)$$

$$\left. e^{i(p \cdot x - p' \cdot y)} p'_i \delta^3(\mathbf{p} - \mathbf{p}') \right\} \quad (24)$$

$$= \frac{i}{2(2\pi)^3} \int d^3 p p_i \left(e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)} \right) \quad (25)$$

We now note that the integrand in the last line is an odd function of p . That is, if we replace p by $-p$ the integrand changes sign. The integral of an odd function over all p space is zero, so we have

$$[\phi(t, \mathbf{x}), \partial_i \phi(t, \mathbf{y})] = 0 \quad (26)$$

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