

REAL SCALAR FIELD - COMMUTATORS

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Post date: 28 May 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.4.

The real scalar field ϕ and its conjugate momentum can be decomposed into Fourier integrals

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \left(a(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x} \right) \quad (1)$$

where

$$E_p = p_0 = +\sqrt{\mathbf{p}^2 + m^2} \quad (2)$$

$$a(p) = \frac{A(p)}{\sqrt{2E_p}} \quad (3)$$

The conjugate momentum for this field is

$$\Pi(x) = \dot{\phi}(x) \quad (4)$$

$$= \frac{i}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{E_p}{2}} \left(-a(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x} \right) \quad (5)$$

We can invert these transforms to get expressions for a and a^\dagger as Fourier integrals over ϕ and Π . We can then use the commutators

$$[\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}) \quad (6)$$

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = 0 \quad (7)$$

$$[\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})] = 0 \quad (8)$$

to calculate the commutators of a and a^\dagger . To invert the transforms, we start by multiplying 1 by $\frac{1}{(2\pi)^{3/2}} e^{ip' \cdot x}$ on both sides and integrating over x :

$$\frac{1}{(2\pi)^{3/2}} \int d^3x e^{ip' \cdot x} \phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{\sqrt{2E_p}} \int d^3x \left(a(p) e^{-ix \cdot (p-p')} + a^\dagger(p) e^{ix \cdot (p+p')} \right) \quad (9)$$

We can use the formula for the delta function:

$$\frac{1}{(2\pi)^3} \int d^3x e^{ix \cdot (\mathbf{p} - \mathbf{p}')} = \delta^3(\mathbf{p} - \mathbf{p}') \quad (10)$$

We get

$$\frac{1}{(2\pi)^{3/2}} \int d^3x e^{ip' \cdot x} \phi(x) = \int \frac{d^3p}{\sqrt{2E_p}} \left(a(p) e^{-it \cdot (p_0 - p'_0)} \delta^3(\mathbf{p} - \mathbf{p}') + a^\dagger(p) e^{it \cdot (p_0 + p'_0)} \delta^3(\mathbf{p} + \mathbf{p}') \right) \quad (11)$$

The component p_0 satisfies the relation

$$E_p = p_0 = +\sqrt{\mathbf{p}^2 + m^2} \quad (12)$$

so the delta functions in 11 force the exponents of the two exponential factors to be zero, so both exponentials become 1. We can then do the integral over p and drop the prime on the remaining p' (since p is just a dummy variable of integration) to get

$$\frac{1}{(2\pi)^{3/2}} \int d^3x e^{ip \cdot x} \phi(x) = \frac{1}{\sqrt{2E_p}} \left(a(p) + a^\dagger(-p) \right) \quad (13)$$

We've used

$$E_{-p} = \sqrt{(-\mathbf{p})^2 + m^2} = \sqrt{\mathbf{p}^2 + m^2} = E_p \quad (14)$$

We can run through a similar calculation with 5 to get

$$\frac{1}{(2\pi)^{3/2}} \int d^3x e^{ip \cdot x} \Pi(x) = i\sqrt{\frac{E_p}{2}} \left(-a(p) + a^\dagger(-p) \right) \quad (15)$$

Combining 13 and 15 we get

$$a(p) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{ip \cdot x} \left(\sqrt{\frac{E_p}{2}} \phi(x) + \frac{i}{\sqrt{2E_p}} \Pi(x) \right) \quad (16)$$

$$a^\dagger(p) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-ip \cdot x} \left(\sqrt{\frac{E_p}{2}} \phi(x) - \frac{i}{\sqrt{2E_p}} \Pi(x) \right) \quad (17)$$

We can now apply the commutators 7 to work out the commutators of a and a^\dagger . Only terms involving the commutator $[\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})]$ will contribute, so we have

$$[a(p), a^\dagger(p')] = \frac{1}{2(2\pi)^3} \left[\int d^3x \int d^3y e^{i(p \cdot x - p' \cdot y)} \left(-i \sqrt{\frac{E_p}{E_{p'}}} [\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] + \right. \right. \quad (18)$$

$$\left. \left. i \sqrt{\frac{E_{p'}}{E_p}} [\Pi(t, \mathbf{x}), \phi(t, \mathbf{y})] \right) \right] \quad (19)$$

$$= \frac{1}{2(2\pi)^3} \left[\int d^3x \int d^3y e^{i(p \cdot x - p' \cdot y)} \left(-i \sqrt{\frac{E_p}{E_{p'}}} [\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] - \right. \right. \quad (20)$$

$$\left. \left. i \sqrt{\frac{E_{p'}}{E_p}} [\phi(t, \mathbf{y}), \Pi(t, \mathbf{x})] \right) \right] \quad (21)$$

$$= \frac{1}{2(2\pi)^3} \left[\int d^3x \int d^3y e^{i(p \cdot x - p' \cdot y)} \left(\sqrt{\frac{E_p}{E_{p'}}} + \sqrt{\frac{E_{p'}}{E_p}} \right) \delta^3(\mathbf{x} - \mathbf{y}) \right] \quad (22)$$

$$= \frac{1}{2(2\pi)^3} \left[\int d^3x e^{i(p-p') \cdot x} \left(\sqrt{\frac{E_p}{E_{p'}}} + \sqrt{\frac{E_{p'}}{E_p}} \right) \right] \quad (23)$$

$$= \frac{1}{2} \left(\sqrt{\frac{E_p}{E_{p'}}} + \sqrt{\frac{E_{p'}}{E_p}} \right) e^{i(p_0 - p'_0) \cdot t} \delta^3(\mathbf{p} - \mathbf{p}') \quad (24)$$

$$= \delta^3(\mathbf{p} - \mathbf{p}') \quad (25)$$

where we used 10 and also 12 to show that if $\mathbf{p} = \mathbf{p}'$, then $E_p = E_{p'}$ and $p_0 = p'_0$ so the exponential becomes 1 in the penultimate line.

The other commutation relations can be calculated similarly. For example

$$[a(p), a(p')] = \frac{1}{2(2\pi)^3} \left[\int d^3x \int d^3y e^{i(p \cdot x + p' \cdot y)} \left(i\sqrt{\frac{E_p}{E_{p'}}} [\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] + \right. \right. \quad (26)$$

$$\left. i\sqrt{\frac{E_{p'}}{E_p}} [\Pi(t, \mathbf{x}), \phi(t, \mathbf{y})] \right) \quad (27)$$

$$= \frac{1}{2(2\pi)^3} \left[\int d^3x \int d^3y e^{i(p \cdot x + p' \cdot y)} \left(i\sqrt{\frac{E_p}{E_{p'}}} [\phi(t, \mathbf{x}), \Pi(t, \mathbf{y})] - \right. \quad (28)$$

$$\left. i\sqrt{\frac{E_{p'}}{E_p}} [\phi(t, \mathbf{y}), \Pi(t, \mathbf{x})] \right) \quad (29)$$

$$= \frac{1}{2(2\pi)^3} \left[\int d^3x \int d^3y e^{i(p \cdot x + p' \cdot y)} \left(\sqrt{\frac{E_p}{E_{p'}}} - \sqrt{\frac{E_{p'}}{E_p}} \right) \delta^3(\mathbf{x} - \mathbf{y}) \right] \quad (30)$$

$$= \frac{1}{2(2\pi)^3} \left[\int d^3x e^{i(p+p') \cdot x} \left(\sqrt{\frac{E_p}{E_{p'}}} - \sqrt{\frac{E_{p'}}{E_p}} \right) \right] \quad (31)$$

$$= \frac{1}{2} \left(\sqrt{\frac{E_p}{E_{p'}}} - \sqrt{\frac{E_{p'}}{E_p}} \right) e^{i(p_0+p'_0) \cdot t} \delta^3(\mathbf{p} - \mathbf{p}') \quad (32)$$

$$= 0 \quad (33)$$

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