REAL SCALAR FIELD - COMMUTATORS

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Post date: 28 May 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.4.

The real scalar field ϕ and its conjugate momentum can be decomposed into Fourier integrals

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2E_p}}\left(a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\right) \tag{1}$$

where

$$E_p=p_0=+\sqrt{\mathbf{p}^2+m^2} \tag{2}$$

$$a(p)=\frac{A(p)}{\sqrt{2E_p}} \tag{3}$$

The conjugate momentum for this field is

$$\Pi(x)=\dot{\phi}(x) \tag{4}$$

$$=\frac{i}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{E_p}{2}}\left(-a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\right) \tag{5}$$

We can invert these transforms to get expressions for a and $a^\dagger$ as Fourier integrals over ϕ and Π. We can then use the commutators

$$[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})]=i\delta^3(\mathbf{x}-\mathbf{y}) \tag{6}$$

$$[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})]=0 \tag{7}$$

$$[\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})]=0 \tag{8}$$

to calculate the commutators of a and $a^\dagger$. To invert the transforms, we start by multiplying 1 by $\frac{1}{(2\pi)^{3/2}}e^{ip'\cdot x}$ on both sides and integrating over x:

$$\frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{ip'\cdot x}\phi(x)=\frac{1}{(2\pi)^3}\int\frac{d^3p}{\sqrt{2E_p}}\int d^3x\left(a(p)\,e^{-ix\cdot(p-p')}+a^\dagger(p)\,e^{ix\cdot(p+p')}\right) \tag{9}$$

We can use the formula for the delta function:

$$\frac{1}{(2\pi)^3}\int d^3x\, e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}=\delta^3\left(\mathbf{p}-\mathbf{p}'\right) \tag{10}$$

We get

$$\frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{ip'\cdot x}\phi(x)=\int\frac{d^3p}{\sqrt{2E_p}}\left(a(p)\,e^{-it\cdot\left(p_0-p_0'\right)}\delta^3\left(\mathbf{p}-\mathbf{p}'\right)+a^\dagger(p)\,e^{it\cdot\left(p_0+p_0'\right)}\delta^3\left(\mathbf{p}+\mathbf{p}'\right)\right) \tag{11}$$

The component p_0 satisfies the relation

$$E_p=p_0=+\sqrt{\mathbf{p}^2+m^2} \tag{12}$$

so the delta functions in 11 force the exponents of the two exponential factors to be zero, so both exponentials become 1. We can then do the integral over p and drop the prime on the remaining p' (since p is just a dummy variable of integration) to get

$$\frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{ip\cdot x}\phi(x)=\frac{1}{\sqrt{2E_p}}\left(a(p)+a^\dagger(-p)\right) \tag{13}$$

We've used

$$E_{-p}=\sqrt{(-\mathbf{p})^2+m^2}=\sqrt{\mathbf{p}^2+m^2}=E_p \tag{14}$$

We can run through a similar calculation with 5 to get

$$\frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{ip\cdot x}\Pi(x)=i\sqrt{\frac{E_p}{2}}\left(-a(p)+a^\dagger(-p)\right) \tag{15}$$

Combining 13 and 15 we get

$$a(p)=\frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{ip\cdot x}\left(\sqrt{\frac{E_p}{2}}\phi(x)+\frac{i}{\sqrt{2E_p}}\Pi(x)\right) \tag{16}$$

$$a^\dagger(p)=\frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{-ip\cdot x}\left(\sqrt{\frac{E_p}{2}}\phi(x)-\frac{i}{\sqrt{2E_p}}\Pi(x)\right) \tag{17}$$

We can now apply the commutators 7 to work out the commutators of a and $a^{\dagger}$. Only terms involving the commutator $[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})]$ will contribute, so we have

$$\left[a(p),a^{\dagger}(p')\right] = \frac{1}{2(2\pi)^3}\left[\int d^3x\int d^3y\, e^{i(p\cdot x-p'\cdot y)}\left(-i\sqrt{\frac{E_p}{E_{p'}}}[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})]+\right.\right. \tag{18}$$

$$\left.\left. i\sqrt{\frac{E_{p'}}{E_p}}[\Pi(t,\mathbf{x}),\phi(t,\mathbf{y})]\right)\right] \tag{19}$$

$$= \frac{1}{2(2\pi)^3}\left[\int d^3x\int d^3y\, e^{i(p\cdot x-p'\cdot y)}\left(-i\sqrt{\frac{E_p}{E_{p'}}}[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})]-\right.\right. \tag{20}$$

$$\left.\left. i\sqrt{\frac{E_{p'}}{E_p}}[\phi(t,\mathbf{y}),\Pi(t,\mathbf{x})]\right)\right] \tag{21}$$

$$= \frac{1}{2(2\pi)^3}\left[\int d^3x\int d^3y\, e^{i(p\cdot x-p'\cdot y)}\left(\sqrt{\frac{E_p}{E_{p'}}}+\sqrt{\frac{E_{p'}}{E_p}}\right)\delta^3(\mathbf{x}-\mathbf{y})\right] \tag{22}$$

$$= \frac{1}{2(2\pi)^3}\left[\int d^3x\, e^{i(p-p')\cdot x}\left(\sqrt{\frac{E_p}{E_{p'}}}+\sqrt{\frac{E_{p'}}{E_p}}\right)\right] \tag{23}$$

$$= \frac{1}{2}\left(\sqrt{\frac{E_p}{E_{p'}}}+\sqrt{\frac{E_{p'}}{E_p}}\right)e^{i\left(p_0-p'_0\right)\cdot t}\delta^3\left(\mathbf{p}-\mathbf{p}'\right) \tag{24}$$

$$= \delta^3\left(\mathbf{p}-\mathbf{p}'\right) \tag{25}$$

where we used 10 and also 12 to show that if $\mathbf{p}=\mathbf{p}'$, then $E_p=E_{p'}$ and $p_0=p'_0$ so the exponential becomes 1 in the penultimate line.

The other commutation relations can be calculated similarly. For example

$$\left[a\left(p\right),a\left(p'\right)\right]=\frac{1}{2\left(2\pi\right)^{3}}\left[\int d^{3}x\int d^{3}y\;e^{i\left(p\cdot x+p'\cdot y\right)}\left(i\sqrt{\frac{E_{p}}{E_{p'}}}\left[\phi\left(t,\mathbf{x}\right),\Pi\left(t,\mathbf{y}\right)\right]+\right.\right. \tag{26}$$

$$\left.\left.i\sqrt{\frac{E_{p'}}{E_{p}}}\left[\Pi\left(t,\mathbf{x}\right),\phi\left(t,\mathbf{y}\right)\right]\right)\right] \tag{27}$$

$$=\frac{1}{2\left(2\pi\right)^{3}}\left[\int d^{3}x\int d^{3}y\;e^{i\left(p\cdot x+p'\cdot y\right)}\left(i\sqrt{\frac{E_{p}}{E_{p'}}}\left[\phi\left(t,\mathbf{x}\right),\Pi\left(t,\mathbf{y}\right)\right]-\right.\right. \tag{28}$$

$$\left.\left.i\sqrt{\frac{E_{p'}}{E_{p}}}\left[\phi\left(t,\mathbf{y}\right),\Pi\left(t,\mathbf{x}\right)\right]\right)\right] \tag{29}$$

$$=\frac{1}{2\left(2\pi\right)^{3}}\left[\int d^{3}x\int d^{3}y\;e^{i\left(p\cdot x+p'\cdot y\right)}\left(\sqrt{\frac{E_{p}}{E_{p'}}}-\sqrt{\frac{E_{p'}}{E_{p}}}\right)\delta^{3}\left(\mathbf{x}-\mathbf{y}\right)\right] \tag{30}$$

$$=\frac{1}{2\left(2\pi\right)^{3}}\left[\int d^{3}x\;e^{i\left(p+p'\right)\cdot x}\left(\sqrt{\frac{E_{p}}{E_{p'}}}-\sqrt{\frac{E_{p'}}{E_{p}}}\right)\right] \tag{31}$$

$$=\frac{1}{2}\left(\sqrt{\frac{E_{p}}{E_{p'}}}-\sqrt{\frac{E_{p'}}{E_{p}}}\right)e^{i\left(p_{0}+p_{0}'\right)\cdot t}\delta^{3}\left(\mathbf{p}-\mathbf{p}'\right) \tag{32}$$

$$=0 \tag{33}$$

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