NORMAL ORDERING

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.5.

The real scalar field ϕ and its conjugate momentum can be decomposed into Fourier integrals

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2E_p}}\left(a(p)e^{-ip\cdot x}+a^{\dagger}(p)e^{ip\cdot x}\right) \tag{1}$$

where

$$E_p=p_0=+\sqrt{\mathbf{p}^2+m^2} \tag{2}$$

$$a(p)=\frac{A(p)}{\sqrt{2E_p}} \tag{3}$$

The conjugate momentum for this field is

$$\Pi(x)=\dot{\phi}(x) \tag{4}$$

$$=\frac{i}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{E_p}{2}}\left(-a(p)e^{-ip\cdot x}+a^{\dagger}(p)e^{ip\cdot x}\right) \tag{5}$$

In section 3.3, L&P derive an expression for the Hamiltonian of a free (no potential) real scalar field in terms of a and $a^{\dagger}$. This involves using the definition equation for the Hamiltonian density, which is

$$\mathscr{H}=\frac{1}{2}\left(\Pi^2+(\nabla\phi)^2+m^2\phi^2\right) \tag{6}$$

then substituting for ϕ and Π using 1 and 5 and integrating over all space to get the total Hamiltonian. This integration introduces some delta functions which allow one of the integrations over momentum space to be done, with the final result

$$H=\frac{1}{2}\int d^3p\,E_p\left[a^{\dagger}(p)a(p)+a(p)a^{\dagger}(p)\right] \tag{7}$$

The problem with this Hamiltonian is that the ground state $|0\rangle$ has infinite energy, as we can see by using the commutation relation

$$\left[a(p), a^{\dagger}(p')\right] = \delta^{3}(\mathbf{p}-\mathbf{p}') \tag{8}$$

Using to invert the last term in 7 we get

The subscript p in $\delta^{3}(\mathbf{0})_{p}$ indicates that the delta function is in momentum space.

$$H = \int d^{3}p\, E_{p}\left(a^{\dagger}(p)\, a(p) + \frac{1}{2}\delta^{3}(\mathbf{0})_{p}\right) \tag{9}$$

If we take the ground state $|0\rangle$ to be normalized and defined so that

$$a(p)|0\rangle = 0$$

for all p, then the expectation value of H in the ground state is

$$\langle 0|H|0\rangle = \frac{1}{2}\delta^{3}(\mathbf{0})_{p}\int d^{3}p\, E_{p} \tag{10}$$

Since $E_{p} \geq 0$ for all p and the delta function is infinite, this result gives an infinite ground state energy. This problem is 'solved' by a process known as *normal ordering*, which involves ignoring the commutation relations 8 for the operators a and $a^{\dagger}$ and moving them so that all annihilation operators (a) are to the right of all creation operators ($a^{\dagger}$). This process results in

The colons on either side of H are the notation for normal ordering.

$$:H: = \int d^{3}p\, E_{p} a^{\dagger}(p)\, a(p) \tag{11}$$

$$\langle 0|:H:|0\rangle = 0 \tag{12}$$

so the ground state now has zero energy. This has all the feeling of a kludge, since there doesn't appear to be any justification for ignoring 8 just in this one case (and then imposing the commutator in all future calculations). In any case, I suppose it works, but I get the feeling that there's something wrong with a theory that requires something like this.

We can extend this calculation to find the components of the four-momentum operator P^{μ} in terms of creation and annihilation operators by following steps similar to those in L&P's section 3.3. The four-momentum is defined in terms of the stress-energy tensor as

$$P^{\nu} = \int d^{3}x\, T^{0\nu} \tag{13}$$

For a free real scalar field, we have

$$T^{\mu\nu} \equiv \partial^{\mu}\phi\partial^{\nu}\phi - g^{\mu\nu}\mathscr{L} \tag{14}$$

so

$$T^{0\nu} = \dot{\phi}\partial^{\nu}\phi - g^{0\nu}\mathscr{L} \tag{15}$$

Since $g^{\mu\nu}$ is diagonal, the spatial components are

$$T^{0i} = \dot{\phi}\partial^{i}\phi \tag{16}$$

The 0 compoment is

$$T^{00} = \dot{\phi}^{2} - \mathscr{L} \tag{17}$$

which is just the Hamiltonian density 6. Thus we've already worked out $P^0 = \int d^3x\,\mathscr{H}$, as it turns out to be 11 (after applying normal ordering).

The three spatial components are a bit simpler. If we substitute 1 into 16 we get

$$T^{0i} = \frac{1}{(2\pi)^3}\int d^3p \frac{iE_p}{\sqrt{2E_p}}\left(-a(p)\,e^{-ip\cdot x} + a^{\dagger}(p)\,e^{ip\cdot x}\right)\times \tag{18}$$

$$\int d^3p' \frac{ip'^i}{\sqrt{2E_{p'}}}\left(-a\left(p'\right)e^{-ip'\cdot x} + a^{\dagger}\left(p'\right)e^{ip'\cdot x}\right) \tag{19}$$

We then need to integrate this entire expression over d^3x. There will be four terms in this integral, so let's take them one at a time. If we take the $a(p)$ term multiplied by the $a(p')$ term, which we'll refer to as 1), we have

$$1 = -\frac{1}{(2\pi)^3}\int d^3x \int d^3p \int d^3p' \frac{E_p}{\sqrt{2E_p}}\frac{p'^i}{\sqrt{2E_{p'}}} a(p)\,a\left(p'\right)e^{-i(p+p')\cdot x} \tag{20}$$

The integral over x gives a delta function, so we get

$$1 = -\int d^3p \int d^3p' \frac{E_p}{\sqrt{2E_p}}\frac{p'^i}{\sqrt{2E_{p'}}} a(p)\,a\left(p'\right)e^{-i\left(E_p+E_{p'}\right)t}\delta^3\left(\mathbf{p}+\mathbf{p}'\right) \tag{21}$$

$$= -\frac{1}{2}\int d^3p\; p^i a(p)\,a(-p)\,e^{-2iE_pt} \tag{22}$$

The integrand is an odd function of p^i so integrating over this component of momentum will give zero. The same argument applies to the term with $a^{\dagger}(p)$ multiplied by $a^{\dagger}(p')$. We are therefore left with the two cross terms. Consider the term with $a^{\dagger}(p)$ multiplied by $a(p')$, which we'll refer to as 2. We have

$$\textcircled{2} = \frac{1}{(2\pi)^3} \int d^3x \int d^3p \int d^3p' \frac{E_p}{\sqrt{2E_p}} \frac{p'^i}{\sqrt{2E_{p'}}} a^\dagger(p)\, a(p')\, e^{-i(p-p')\cdot x} \tag{23}$$

$$= \int d^3p \int d^3p' \frac{E_p}{\sqrt{2E_p}} \frac{p'^i}{\sqrt{2E_{p'}}} a^\dagger(p)\, a(p')\, e^{-i\left(E_p - E_{p'}\right)t} \delta^3\left(\mathbf{p} - \mathbf{p}'\right) \tag{24}$$

$$= \frac{1}{2} \int d^3p\; p^i a^\dagger(p)\, a(p) \tag{25}$$

This integrand is neither even nor odd, so is not, in general, zero.

The other term 3 can be worked out similarly to get

$$\textcircled{3} = \frac{1}{(2\pi)^3} \int d^3x \int d^3p \int d^3p' \frac{E_p}{\sqrt{2E_p}} \frac{p'^i}{\sqrt{2E_{p'}}} a(p)\, a^\dagger(p')\, e^{-i(p-p')\cdot x} \tag{26}$$

$$= \int d^3p \int d^3p' \frac{E_p}{\sqrt{2E_p}} \frac{p'^i}{\sqrt{2E_{p'}}} a(p)\, a^\dagger(p')\, e^{-i\left(E_p - E_{p'}\right)t} \delta^3\left(\mathbf{p} - \mathbf{p}'\right) \tag{27}$$

$$= \frac{1}{2} \int d^3p\; p^i a(p)\, a^\dagger(p) \tag{28}$$

We therefore have

$$P^i = \textcircled{2} + \textcircled{3} \tag{29}$$

$$= \frac{1}{2} \int d^3p\; p^i \left(a^\dagger(p)\, a(p) + a(p)\, a^\dagger(p)\right) \tag{30}$$

We can apply the commutator 8 to this to get

$$P^i = \int d^3p\; p^i \left(a^\dagger(p)\, a(p) + \frac{1}{2}\delta^3(\mathbf{0})_p\right) \tag{31}$$

In this case, the term involving the delta function integrates to zero, because the factor of p^i makes the integrand an odd function. Thus there is no need for normal ordering in this case. We therefore have

$$P^i = \int d^3p\; p^i a^\dagger(p)\, a(p) \tag{32}$$

We can also work out the commutator $\left[\phi, P_\mu\right]$. From 1 we have, using 8:

$$\left[\phi,P_\mu\right]=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p\,d^3p'}{\sqrt{2E_p}}p'_\mu\left(e^{-ip\cdot x}\left[a(p),a^\dagger(p')a(p')\right]+\right.\tag{33}$$

$$\left.e^{ip\cdot x}\left[a^\dagger(p),a^\dagger(p')a(p')\right]\right)\tag{34}$$

$$=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p\,d^3p'}{\sqrt{2E_p}}p'_\mu\left(e^{-ip\cdot x}\left[a(p),a^\dagger(p')\right]a(p')+\right.\tag{35}$$

$$\left.e^{ip\cdot x}a^\dagger(p')\left[a^\dagger(p),a(p')\right]\right)\tag{36}$$

$$=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p\,d^3p'}{\sqrt{2E_p}}p'_\mu\left(e^{-ip\cdot x}\left[a(p),a^\dagger(p')\right]a(p')-\right.\tag{37}$$

$$\left.e^{ip\cdot x}a^\dagger(p')\left[a(p'),a^\dagger(p)\right]\right)\tag{38}$$

$$=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p\,d^3p'}{\sqrt{2E_p}}p'_\mu\delta^3\left(\mathbf{p}-\mathbf{p}'\right)\left(e^{-ip\cdot x}a(p')-e^{ip\cdot x}a^\dagger(p')\right)\tag{39}$$

$$=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2E_p}}p_\mu\left(e^{-ip\cdot x}a(p)-e^{ip\cdot x}a^\dagger(p)\right)\tag{40}$$

$$=i\partial_\mu\phi\tag{41}$$

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