FOCK SPACE - NUMBER OPERATOR

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.6.

For the real scalar field, the vacuum state $|0\rangle$ is defined as the normalized state containing zero particles which gives zero when operated on by any annihilation operator, that is

$$a(p)|0\rangle = 0 \tag{1}$$

for any momentum p. To create a particle, we apply the creation operator, so that

$$|p\rangle = a^{\dagger}(p)|0\rangle \tag{2}$$

is a state containing a single particle with momentum p. Using the commutation relation

$$\left[a(p), a^{\dagger}(p')\right] = \delta^3(\mathbf{p}-\mathbf{p}') \tag{3}$$

we have

$$\langle p'|p\rangle = \left\langle 0\left|a(p')\,a^{\dagger}(p)\right|0\right\rangle \tag{4}$$

$$= \langle 0|\delta^3(\mathbf{p}-\mathbf{p}')|0\rangle + \left\langle 0\left|a^{\dagger}(p)\,a(p')\right|0\right\rangle \tag{5}$$

$$= \delta^3(\mathbf{p}-\mathbf{p}') + 0 \tag{6}$$

$$= \delta^3(\mathbf{p}-\mathbf{p}') \tag{7}$$

The normalized state with several particles of various momenta is given by

$$|n_1, n_2, \ldots, n_i, \ldots\rangle = \frac{1}{\sqrt{n_1!n_2!\ldots n_i!\ldots}}\left(\hat{a}_1^{\dagger}\right)^{n_1}\left(\hat{a}_2^{\dagger}\right)^{n_2}\ldots\left(\hat{a}_i^{\dagger}\right)^{n_i}\ldots|0\rangle \tag{8}$$

where $|n_1, n_2, \ldots, n_i, \ldots\rangle$ is the state with n_i particles with momentum p_i. The set of all such states (including the vacuum) is called *Fock space,* named after Vladimir Fock (1898 - 1974), a Soviet physicist.

In Fock space, we can define a number operator as

$$\mathcal{N}=\int d^3p\,a^\dagger(p)\,a(p) \tag{9}$$

To see that this operator counts the number of particles in a multiparticle vector, consider the state with a single particle in each of N different momentum states, that is $|p_1,p_2,\ldots,p_N\rangle$. We have

$$|p_1,p_2,\ldots,p_N\rangle=a^\dagger(p_1)\,a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{10}$$

We can now apply the number operator and see the results using 3.

$$\mathcal{N}|p_1,p_2,\ldots,p_N\rangle=\int d^3p\,a^\dagger(p)\,a(p)\,a^\dagger(p_1)\,a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{11}$$

$$=\int d^3p\,a^\dagger(p)\left[\delta^3(\mathbf{p}-\mathbf{p}_1)+a^\dagger(p_1)\,a(p)\right]a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{12}$$

$$=a^\dagger(p_1)\,a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle+ \tag{13}$$

$$\int d^3p\,a^\dagger(p)\,a^\dagger(p_1)\,a(p)\,a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{14}$$

$$=|p_1,p_2,\ldots,p_N\rangle+\int d^3p\,a^\dagger(p)\,a^\dagger(p_1)\,a(p)\,a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{15}$$

$$=|p_1,p_2,\ldots,p_N\rangle+\int d^3p\,a^\dagger(p_1)\,a^\dagger(p)\,a(p)\,a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{16}$$

The last line uses the fact that $\left[a^\dagger(p),a^\dagger(p_1)\right]=0$ to swap the first two creation operators in the integral. We can now apply the commutator to the terms $a(p)\,a^\dagger(p_2)$ in the integral on the last line, which yields another term of $|p_1,p_2,\ldots,p_N\rangle$ plus another integral with the $a^\dagger(p)\,a(p)$operator shifted another slot to the right. The process will continue N times until the $a(p)$ comes up against the $|0\rangle$, so the final integral will be zero because of 1. Thus the final result is

$$\mathcal{N}|p_1,p_2,\ldots,p_N\rangle=N|p_1,p_2,\ldots,p_N\rangle \tag{17}$$

This argument doesn't actually rely on the momenta of the particles being all different, so we could apply the same reasoning to a general state 8 to see that $\mathcal{N}$ always counts the number of particles in a given state.

We can also work out the commutators of $\mathscr{N}$ with the creation and annihilation operators. We have

$$\left[\mathscr{N},a^{\dagger}(k)\right]=\int d^{3}p\,a^{\dagger}(p)\left[a(p),a^{\dagger}(k)\right] \tag{18}$$

$$=\int d^{3}p\,a^{\dagger}(p)\,\delta^{3}(\mathbf{p}-\mathbf{k}) \tag{19}$$

$$=a^{\dagger}(k) \tag{20}$$

$$[\mathscr{N},a(k)]=\int d^{3}p\,\left[a^{\dagger}(p),a(k)\right]a(p) \tag{21}$$

$$=-\int d^{3}p\,\left[a(p),a^{\dagger}(k)\right]a(p) \tag{22}$$

$$=-\int d^{3}p\,\delta^{3}(\mathbf{p}-\mathbf{k})\,a(p) \tag{23}$$

$$=-a(k) \tag{24}$$

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