COMPLEX SCALAR FIELD - QUANTIZATION, PARTICLES AND ANTIPARTICLES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.7.

We now consider the complex scalar field with Lagrangian (for a free particle with $V=0$):

$$\mathscr{L}=\left(\partial_{\mu}\phi\right)^{\dagger}\left(\partial^{\mu}\phi\right)-m^{2}\phi^{\dagger}\phi \tag{1}$$

We can write the field in terms of its real and imaginary parts as

$$\phi\left(x\right)=\frac{1}{\sqrt{2}}\left(\phi_{1}\left(x\right)+i\phi_{2}\left(x\right)\right) \tag{2}$$

where ϕ_1 and ϕ_2 are both real fields. We can decompose each of these fields into a Fourier integral using the same technique as for the real scalar field. This introduces the creation and annihilation operators for each component of the field, which obey the commutation relations

$$\left[a_{1}\left(p\right),a_{1}^{\dagger}\left(p'\right)\right]=\left[a_{2}\left(p\right),a_{2}^{\dagger}\left(p'\right)\right]=\delta^{3}\left(\mathbf{p}-\mathbf{p}'\right) \tag{3}$$

with all other commutators being zero.

It turns out to be more physically meaningful to define different creation and annihilation operators which are linear combinations of a_1 and a_2. The annihilation operators are

$$a\left(p\right)\equiv\frac{1}{\sqrt{2}}\left(a_{1}\left(p\right)+ia_{2}\left(p\right)\right) \tag{4}$$

$$\widehat{a}\left(p\right)\equiv\frac{1}{\sqrt{2}}\left(a_{1}\left(p\right)-ia_{2}\left(p\right)\right) \tag{5}$$

with corresponding creation operators

$$a^{\dagger}(p) \equiv \frac{1}{\sqrt{2}}\left(a_1^{\dagger}(p) - i a_2^{\dagger}(p)\right) \tag{6}$$

$$\widehat{a}(p) \equiv \frac{1}{\sqrt{2}}\left(a_1^{\dagger}(p) + i a_2^{\dagger}(p)\right) \tag{7}$$

The fields can now be written in terms of these operators as

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left(a(p)e^{-ip\cdot x} + \widehat{a}^{\dagger}(p)e^{ip\cdot x}\right) \tag{8}$$

$$\phi^{\dagger}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left(\widehat{a}(p)e^{-ip\cdot x} + a^{\dagger}(p)e^{ip\cdot x}\right) \tag{9}$$

Using 3, these new operators obey the commutation relations

$$\left[a(p), a^{\dagger}(p')\right] = \left[\widehat{a}(p), \widehat{a}^{\dagger}(p')\right] = \delta^3(\mathbf{p} - \mathbf{p}') \tag{10}$$

The complex scalar field Lagrangian is invariant under the transformation

$$\phi \to e^{-iq\theta}\phi \tag{11}$$

$$\phi^{\dagger} \to e^{iq\theta}\phi^{\dagger} \tag{12}$$

We've seen that applying Noether's theorem to this invariance leads to the conserved current

$$j^{\mu} = iq\left(\phi^{\dagger}\partial^{\mu}\phi - \phi\partial^{\mu}\phi^{\dagger}\right) \tag{13}$$

with a corresponding conserved charge Q given by

$$Q = \int d^3x\, j^0 \tag{14}$$

L&P simply state this charge in terms of a and $\widehat{a}$, but a derivation is useful to see, so here we go. We have

$$Q = iq\int d^3x\left(\phi^{\dagger}\partial^0\phi - \phi\partial^0\phi^{\dagger}\right) \tag{15}$$

so we need to substitute for the fields using 8 and 9. The derivatives are

$$\partial^0\phi = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} iE_{p'} \left(-a\left(p'\right)e^{-ip'\cdot x} + \widehat{a}^\dagger\left(p'\right)e^{ip'\cdot x}\right) \tag{16}$$

$$\partial_0\phi^\dagger = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} iE_{p'} \left(-\widehat{a}\left(p'\right)e^{-ip'\cdot x} + a^\dagger\left(p'\right)e^{ip'\cdot x}\right) \tag{17}$$

Each of the two terms in the integrand of 15 gives rise to four terms when the fields are multiplied out. The integral over x gives a delta function over $\mathbf{p}-\mathbf{p}'$ in each term, so one of the integrals over momentum can be done. For the first term, we have

$$\int d^3x \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(\widehat{a}(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right) \times \tag{18}$$

$$\int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} iE_{p'} \left(-a\left(p'\right)e^{-ip'\cdot x} + \widehat{a}^\dagger\left(p'\right)e^{ip'\cdot x}\right) = \tag{19}$$

$$\int \frac{d^3p}{2} i\left(\widehat{a}(p)a(-p)e^{-2iE_pt} + a^\dagger(p)\widehat{a}^\dagger(-p)e^{2iE_pt} + \widehat{a}(p)\widehat{a}^\dagger(p) - a^\dagger(p)a(p)\right) \tag{20}$$

The second term in 15 gives

$$\int d^3x \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(a(p)e^{-ip\cdot x} + \widehat{a}^\dagger(p)e^{ip\cdot x}\right) \times \tag{21}$$

$$\int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} iE_{p'} \left(-\widehat{a}\left(p'\right)e^{-ip'\cdot x} + a^\dagger\left(p'\right)e^{ip'\cdot x}\right) = \tag{22}$$

$$\int \frac{d^3p}{2} i\left(a(p)\widehat{a}(-p)e^{-2iE_pt} + \widehat{a}^\dagger(p)a^\dagger(-p)e^{2iE_pt} - \widehat{a}^\dagger(p)\widehat{a}(p) + a(p)a^\dagger(p)\right) \tag{23}$$

We can change the integration variable from $\mathbf{p}$ to $-\mathbf{p}$ in the first two terms of 23 (this changes the sign of d^3p, but there are also 3 changes of sign when we reverse the order of integration in the 3 components of $\mathbf{p}$, so the sign changes cancel out). As a result, the first two terms in 23 cancel the first two terms in 20 when we take the difference in 15. The remaining two terms in 20 and 23 don't cancel when we take the difference, so we get

$$Q = \frac{iq}{2}\int d^3p\, i\left(\widehat{a}(p)\widehat{a}^\dagger(p) - a^\dagger(p)a(p) + \widehat{a}^\dagger(p)\widehat{a}(p) - a(p)a^\dagger(p)\right) \tag{24}$$

$$= \frac{q}{2}\int d^3p\,\left(-\widehat{a}(p)\widehat{a}^\dagger(p) + a^\dagger(p)a(p) - \widehat{a}^\dagger(p)\widehat{a}(p) + a(p)a^\dagger(p)\right) \tag{25}$$

Using 10, we have

$$Q = \frac{q}{2}\int d^3p\left(2a^\dagger(p)a(p) + \delta^3(\mathbf{0}) - 2\widehat{a}^\dagger(p)\widehat{a}(p) - \delta^3(\mathbf{0})\right) \tag{26}$$

$$= q\int d^3p\left(a^\dagger(p)a(p) - \widehat{a}^\dagger(p)\widehat{a}(p)\right) \tag{27}$$

where the last step assumes it's valid to cancel the two (infinite) delta functions.

The interpretation of 27 is that the operators $a^\dagger$ and a create and annihilate particles, while $\widehat{a}^\dagger$ and $\widehat{a}$ create and annihilate antiparticles (which have opposite charge). Since the operator

$$\mathscr{N} = \int d^3p\, a^\dagger(p)a(p) \tag{28}$$

is the number operator for particles and

$$\widehat{\mathscr{N}} = \int d^3p\, \widehat{a}^\dagger(p)\widehat{a}(p) \tag{29}$$

is the number operator for antiparticles, the operator Q gives the net charge in a system of particles and antiparticles.

If we try to express Q in terms of the original operators a_1 and a_2 we find that Q doesn't have such a simple form. We have (all operators have momentum p):

$$a^{\dagger}a = \frac{1}{2}\left(a_1^{\dagger} - ia_2^{\dagger}\right)(a_1 + ia_2) \tag{30}$$

$$= \frac{1}{2}\left(a_1^{\dagger}a_1 + a_2^{\dagger}a_2 + i\left(a_1^{\dagger}a_2 - a_2^{\dagger}a_1\right)\right) \tag{31}$$

$$\widehat{a}^{\dagger}\widehat{a} = \frac{1}{2}\left(a_1^{\dagger} + ia_2^{\dagger}\right)(a_1 - ia_2) \tag{32}$$

$$= \frac{1}{2}\left(a_1^{\dagger}a_1 + a_2^{\dagger}a_2 - i\left(a_1^{\dagger}a_2 - a_2^{\dagger}a_1\right)\right) \tag{33}$$

$$a^{\dagger}a - \widehat{a}^{\dagger}\widehat{a} = i\left(a_1^{\dagger}a_2 - a_2^{\dagger}a_1\right) \tag{34}$$

$$Q = iq\int d^3p\left(a_1^{\dagger}a_2 - a_2^{\dagger}a_1\right) \tag{35}$$

Thus the operators get jumbled up when used to calculate the charge.

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