COMPLEX SCALAR FIELD - HAMILTONIAN

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.8.

We'll look at deriving the Hamiltonian for the complex scalar field with Lagrangian (for a free particle with $V=0$):

$$\mathscr{L}=\left(\partial_{\mu}\phi\right)^{\dagger}\left(\partial^{\mu}\phi\right)-m^{2}\phi^{\dagger}\phi \tag{1}$$

Our ultimate goal is to find the Hamiltonian in terms of the creation and annihilation operators $\widehat{a}^{\dagger}$, $a^{\dagger}$, $\widehat{a}$ and a. The standard recipe for the Hamiltonian in terms of the fields and Lagrangian is

$$H=\int d^{3}x\,\mathscr{H} \tag{2}$$

where

$$\mathscr{H}=\Pi_{A}\dot{\phi}^{A}-\mathscr{L} \tag{3}$$

where there is an implied sum over the field index A. In the complex scalar field, there are two independent fields which are taken to be ϕ and $\phi^{\dagger}$. The conjugate momentum is defined as

$$\Pi=\frac{\partial\mathscr{L}}{\partial\dot{\phi}}=\dot{\phi}^{\dagger} \tag{4}$$

$$\Pi^{\dagger}=\dot{\phi} \tag{5}$$

The Hamiltonian is therefore

$$\mathscr{H}=2\dot{\phi}^{\dagger}\dot{\phi}-\mathscr{L} \tag{6}$$

$$=\dot{\phi}^{\dagger}\dot{\phi}+\nabla\phi^{\dagger}\cdot\nabla\phi+m^{2}\phi^{\dagger}\phi \tag{7}$$

To convert this to an expression in terms of $\widehat{a}^{\dagger}$, $a^{\dagger}$, $\widehat{a}$ and a, we use the expressions for the fields

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(a(p) e^{-ip\cdot x} + \widehat{a}^\dagger(p) e^{ip\cdot x} \right) \tag{8}$$

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(\widehat{a}(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x} \right) \tag{9}$$

The procedure involves integrating 7 over space, which introduces delta functions in momentum, which can then be used to do one of the momentum integrals. This uses the formula

$$\delta^3(\mathbf{p}-\mathbf{p}') = \frac{1}{(2\pi)^3} \int d^3x\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \tag{10}$$

First, we need the derivatives of ϕ:

$$\dot{\phi} = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} iE_p \left(-a(p) e^{-ip\cdot x} + \widehat{a}^\dagger(p) e^{ip\cdot x} \right) \tag{11}$$

$$\dot{\phi}^\dagger = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} iE_{p'} \left(-\widehat{a}(p') e^{-ip'\cdot x} + a^\dagger(p') e^{ip'\cdot x} \right) \tag{12}$$

We can integrate the first term in 7 over d^3x by multiplying these two expressions together and then doing the integral. We have, after doing the space integral and then using the resulting delta function to do one of the momentum integrals:

$$\int d^3x\, \dot{\phi}^\dagger \dot{\phi} = \frac{1}{2E_p} \int d^3p\, E_p^2 \left[-a(p)\widehat{a}(-p) e^{-2iE_pt} - \right. \tag{13}$$

$$\left. \widehat{a}^\dagger(p) a^\dagger(-p) e^{2iE_pt} + a(p) a^\dagger(p) + \widehat{a}^\dagger(p) \widehat{a}(p) \right] \tag{14}$$

Next, we have

$$\nabla\phi = \int \frac{d^3p\; i\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}} \left(-a(p) e^{-ip\cdot x} + \widehat{a}^\dagger(p) e^{ip\cdot x} \right) \tag{15}$$

$$\nabla\phi^\dagger = \int \frac{d^3p'\; i\mathbf{p}'}{\sqrt{(2\pi)^3 2E_{p'}}} \left(-\widehat{a}(p') e^{-ip'\cdot x} + a^\dagger(p') e^{ip'\cdot x} \right) \tag{16}$$

Doing the space integral and the integral over d^3p' we have

$$\int d^3x\nabla\phi^\dagger\cdot\nabla\phi = \frac{1}{2E_p}\int d^3p\ \left[-\mathbf{p}\cdot(-\mathbf{p})\,a(p)\,\widehat{a}(-p)\,e^{-2iE_pt}-\right. \tag{17}$$

$$\left.\mathbf{p}\cdot(-\mathbf{p})\,\widehat{a}^\dagger(p)\,a^\dagger(-p)\,e^{2iE_pt}+\mathbf{p}^2\left(a(p)\,a^\dagger(p)+\widehat{a}^\dagger(p)\,\widehat{a}(p)\right)\right] \tag{18}$$

$$= \frac{1}{2E_p}\int d^3p\ \mathbf{p}^2\left[a(p)\,\widehat{a}(-p)\,e^{-2iE_pt}+\right. \tag{19}$$

$$\left.\widehat{a}^\dagger(p)\,a^\dagger(-p)\,e^{2iE_pt}+a(p)\,a^\dagger(p)+\widehat{a}^\dagger(p)\,\widehat{a}(p)\right] \tag{20}$$

Finally, the last term in 7 gives

$$\int d^3x\ m^2\phi^\dagger\phi = \frac{1}{2E_p}\int d^3p\ m^2\left[a(p)\,\widehat{a}(-p)\,e^{-2iE_pt}+\right. \tag{21}$$

$$\left.\widehat{a}^\dagger(p)\,a^\dagger(-p)\,e^{2iE_pt}+a(p)\,a^\dagger(p)+\widehat{a}^\dagger(p)\,\widehat{a}(p)\right] \tag{22}$$

Using

$$E_p = \sqrt{\mathbf{p}^2+m^2} \tag{23}$$

we can add up 14, 20 and 22 to get

$$H = \int d^3p\ E_p\left(a(p)\,a^\dagger(p)+\widehat{a}^\dagger(p)\,\widehat{a}(p)\right) \tag{24}$$

To avoid the problem of an infinite ground state energy, we can apply normal ordering to this result to get

$$:H:=\int d^3p\ E_p\left(a^\dagger(p)\,a(p)+\widehat{a}^\dagger(p)\,\widehat{a}(p)\right) \tag{25}$$