FEYNMAN PROPAGATOR FOR A SCALAR FIELD (LAHIRI & PAL)

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Post date: 3 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.9.

We'll revisit the notion of a propagator for a scalar field, which turns out to be useful in describing particle interactions later on. The idea is to start with the Klein-Gordon equation with a source term, that is

$$\left(\Box + m^2\right)\phi(x) = J(x) \tag{1}$$

where $J(x)$ is the source term. The propagator (also known as a Green's function) $G(x-x')$ which satisfies the equation

$$\left(\Box_x + m^2\right)G\left(x-x'\right) = -\delta^4\left(x-x'\right) \tag{2}$$

The subscript x on $\Box_x$ indicates a derivative with respect to x, not x'.

L&P show that if we introduce the Fourier transform of $G(x-x')$ in the form

$$G\left(x-x'\right) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} G(p) \tag{3}$$

then we get

$$G(p) = \frac{1}{p^2-m^2} = \frac{1}{(p^0)^2 - E_p^2} \tag{4}$$

Note that here, all four components of p are independent, so that the usual constraint $p^0 = E_p = \sqrt{\mathbf{p}^2+m^2}$ does not apply. The problem with this solution is that, if we try to use it in the integral 3 by integrating p^0 along the real axis, the denominator of $G(p)$ has a couple of zeroes which we can't integrate through. The traditional way of solving this problem was introduced by Feynman, and gives the Feynman propagator. The idea is to convert the integral to a contour integral in p-space by adding an infinitesimal imaginary quantity to the denominator, so we get, instead of $G(p)$,

$$\Delta_F(p) = \frac{1}{p^2-m^2+i\varepsilon'} = \frac{1}{(p^0)^2 - (E_p - i\varepsilon)^2} \tag{5}$$

where ε and ε', related by

$$\varepsilon' = 2E_p\varepsilon \tag{6}$$

are infinitesimal, and the limit $\varepsilon \to 0$ is taken at the end of the calculation. The Feynman propagator in position space is obtained by replacing $G(p)$ by $\Delta_F(p)$ in 3, so we get

$$\Delta_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{p^2-m^2+i\varepsilon'} \tag{7}$$

L&P show, in their equations 3.70 through 3.76, that through complex analysis, this integral can be converted to an integral in 3-d momentum space, which turns out to be

$$i\Delta_F(x-x') = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\Theta(t-t') e^{-ip\cdot(x-x')} + \Theta(t'-t) e^{ip\cdot(x-x')}\right] \tag{8}$$

where

$$\Theta(t-t') = \begin{cases} 1 & \text{if } t > t' \\ \frac{1}{2} & \text{if } t = t' \\ 0 & \text{if } t < t' \end{cases} \tag{9}$$

This result is obtained by taking the limit $\varepsilon \to 0$ when doing the integral over p^0.

In their problem 3.9, L&P ask us to show that $\Delta_F(x-x')$ satisfies the equation

$$\left(\Box_x + m^2\right)\Delta_F(x-x') = -\delta^4(x-x') \tag{10}$$

This follows directly from 7, since

$$\begin{aligned}
\left(\Box_x + m^2\right)\Delta_F(x-x') &= \lim_{\varepsilon\to 0}\int \frac{d^4p}{(2\pi)^4} \frac{\left(\Box_x + m^2\right) e^{-ip\cdot(x-x')}}{p^2-m^2+i\varepsilon'} && (11)\\
&= \lim_{\varepsilon\to 0}\int \frac{d^4p}{(2\pi)^4} \frac{\left(-p^2 + m^2\right) e^{-ip\cdot(x-x')}}{p^2-m^2+i\varepsilon'} && (12)\\
&= -\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} && (13)\\
&= -\delta^4(x-x') && (14)
\end{aligned}$$

It's not obvious how this result can be derived directly from 8, although since we've shown that 8 follows from 7, the result would appear to be valid.

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