DIRAC EQUATION: ANGULAR MOMENTUM

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Post date: 7 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Eqns 4.38 & 4.39.

The angular momentum operators can be used to generate the infinitesimal transformations

$$\psi'(x)=\left(1-\frac{i}{2}J_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{1}$$

where the $\omega^{\mu\nu}$ are the infinitesimal components of a Lorentz transformation. Using the notation in L&P's section 2.4 we have

- $\psi(x)$ is the function ψ at spacetime point x.
- $\psi'(x)$ is the transformation of the function ψ at the *same* point x.
- $\psi'(x')$ is the transformation of the function ψ at the transformed point x'.

From L&P's equation 2.41, we have

$$\psi'(x)-\psi(x)=\psi'\left(x'\right)-\psi(x)-\partial_\mu\psi'(x)\,\delta x^\mu \tag{2}$$

or

$$\psi'\left(x'\right)=\psi'(x)+\partial_\mu\psi'(x)\,\delta x^\mu \tag{3}$$

For an infinitesimal Lorentz transformation

$$\begin{aligned}
\Lambda_{\mu\nu}&=g_{\mu\nu}+\omega_{\mu\nu} &(4)\\
x'^\mu&=\Lambda^\mu{}_\nu x^\nu &(5)\\
&=x^\mu+\omega^\mu{}_\nu x^\nu &(6)\\
\delta x^\mu&=x'^\mu-x^\mu &(7)\\
&=\omega^\mu{}_\nu x^\nu &(8)\\
&=\omega^{\mu\nu}x_\nu &(9)
\end{aligned}$$

We therefore have

$$\psi'\left(x'\right)=\psi'\left(x\right)+\omega^{\mu\nu}x_{\nu}\partial_{\mu}\psi'\left(x\right) \tag{10}$$

We can now apply the transformation 1 to this equation, and keep only terms up to first order in $\omega^{\mu\nu}$:

$$\psi'\left(x'\right)=\left(1-\frac{i}{2}J_{\mu\nu}\omega^{\mu\nu}\right)\psi\left(x\right)+\omega^{\mu\nu}x_{\nu}\partial_{\mu}\psi\left(x\right) \tag{11}$$

However, we also know that the LHS, under an infinitesimal Lorentz transformation, has the form

$$\psi'\left(x'\right)=S\left(\Lambda\right)\psi\left(x\right) \tag{12}$$

$$=\left(1-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi\left(x\right) \tag{13}$$

Equating the terms in $\omega^{\mu\nu}$ on both sides, we get

$$\left[-\frac{i}{2}J_{\mu\nu}+x_{\nu}\partial_{\mu}+\frac{i}{4}\sigma_{\mu\nu}\right]\omega^{\mu\nu}=0 \tag{14}$$

The transformations $\omega^{\mu\nu}$ are arbitrary, but subject to the condition that $\omega^{\mu\nu}=-\omega^{\nu\mu}$. Therefore, we can swap the indexes $\mu\leftrightarrow\nu$ in this equation (and use the antisymmetry of $J_{\mu\nu}$ and $\sigma_{\mu\nu}$) to get

$$\left[-\frac{i}{2}J_{\nu\mu}+x_{\mu}\partial_{\nu}+\frac{i}{4}\sigma_{\nu\mu}\right]\omega^{\nu\mu}=-\left[-\frac{i}{2}J_{\mu\nu}-x_{\mu}\partial_{\nu}+\frac{i}{4}\sigma_{\mu\nu}\right]\omega^{\mu\nu} \tag{15}$$

Subtracting the RHS from 14, we can now equate the coefficient of $\omega^{\mu\nu}$ to zero to we get

$$-iJ_{\mu\nu}+x_{\nu}\partial_{\mu}-x_{\mu}\partial_{\nu}+\frac{i}{2}\sigma_{\mu\nu}=0 \tag{16}$$

or

$$J_{\mu\nu}=i\left(x_{\mu}\partial_{\nu}-x_{\nu}\partial_{\mu}\right)+\frac{1}{2}\sigma_{\mu\nu} \tag{17}$$

The first term on the RHS is the traditional orbital angular momentum operator, and the second term represents the spin.