DIRAC CREATION AND ANNIHILATION OPERATORS AS FOURIER TRANSFORMS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4.

The Dirac field operator $\psi(x)$ is written in terms of spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ and creation and annihilation operators as a Fourier expansion:

$$\psi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + \hat{f}_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ip\cdot x} \right) \tag{1}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s^\dagger(\mathbf{p}) \overline{u}_s(\mathbf{p}) e^{ip\cdot x} + \hat{f}_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) e^{-ip\cdot x} \right) \tag{2}$$

where $f_s^\dagger$ and f_s are the creation and annihilation operators for particles and $\hat{f}_s^\dagger$ and $\hat{f}_s$ are for antiparticles.

In a similar way to that for scalar fields, we can invert these relations to get expressions for the creation and annihilation operators. We'll need the normalization conditions for the spinors

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = 2E_p \delta_{rs} \tag{3}$$

$$v_r^\dagger(\mathbf{p}) u_s(-\mathbf{p}) = u_r^\dagger(\mathbf{p}) v_s(-\mathbf{p}) = 0 \tag{4}$$

First, we can multiply 1 on the left by $\frac{1}{\sqrt{2(2\pi)^3 E_{p'}}} u_r^\dagger(\mathbf{p}') e^{ip'\cdot x}$ and integrate over x.

$$\frac{1}{\sqrt{2(2\pi)^3 E_{p'}}}\int d^3x\, u_r^\dagger(\mathbf{p}')\, e^{ip'\cdot x}\psi(x) = \frac{1}{2(2\pi)^3}\int d^3x\int d^3p \frac{1}{\sqrt{E_p E_{p'}}} \times \sum_{s=1,2}\Big(f_s(\mathbf{p})\, u_r^\dagger(\mathbf{p}')\, u_s(\mathbf{p})\, e^{i(p'-p)\cdot x} \hat{f}_s^\dagger(\mathbf{p})\, u_r^\dagger(\mathbf{p}')\, v_s(\mathbf{p})\, e^{i(p'+p)\cdot x}\Big) \tag{5}$$

The integrals over x give delta functions in momentum, so we can then do the momentum integral to get

$$\frac{1}{\sqrt{2(2\pi)^3 E_{p'}}}\int d^3x\, u_r^\dagger(\mathbf{p}')\, e^{ip'\cdot x}\psi(x) = \frac{1}{2E_p}\sum_{s=1,2}\Big(f_s(\mathbf{p}')\, u_r^\dagger(\mathbf{p}')\, u_s(\mathbf{p}') \hat{f}_s^\dagger(-\mathbf{p}')\, u_r^\dagger(\mathbf{p}')\, v_s(-\mathbf{p}')\Big) \tag{6}$$

Applying 3 and 4 we see that the second term vanishes and the first reduces to $2E_p\sum_{s=1,2}\delta_{rs}f_s(\mathbf{p}) = 2E_p f_r(\mathbf{p})$ so we get (dropping the prime on $\mathbf{p}$ in the final answer):

$$f_r(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\, u_r^\dagger(\mathbf{p})\,\psi(x)\, e^{ip\cdot x} \tag{7}$$

Notice that $u^\dagger$ is a 4-component row vector and ψ is a 4-component column vector, so the result is a single-component object, not a multi-component vector.

Using similar reasoning, we have

$$\hat{f}_r(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\, \psi^\dagger(x)\, v_r(\mathbf{p})\, e^{ip\cdot x} \tag{8}$$

$$f_r^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\, \psi^\dagger(x)\, u_r(\mathbf{p})\, e^{-ip\cdot x} \tag{9}$$

$$\hat{f}_r^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\, v_r^\dagger(\mathbf{p})\,\psi(x)\, e^{-ip\cdot x} \tag{10}$$

In these formulas, the spinors u and v could be taken outside the integrals since they don't depend on the integration variable x. However, we must retain the order of multiplication, since we're dealing with matrix multiplication. The daggered item (whether it's a spinor or a field operator) must always be on the left, as it's a row vector, and the undaggered item is always on the right, as it's a column vector.

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