ENERGY PROJECTION OPERATORS IN THE DIRAC EQUATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4.

A couple of operators acting on the spinors of the Dirac equation can be defined by

$$\Lambda_{\pm}(\mathbf{p}) \equiv \frac{\pm\not{p}+m}{2m} \tag{1}$$

We can see the effects of these operators as follows. Our goal is to show is that they are projection operators that project out states of positive or negative energy. We need the result

$$\not{p}\not{p} = p^2 \tag{2}$$

Any projection operator should have the same effect on a state, no matter how many times it is applied. That is, for a state $|\psi\rangle$ a projection operator P should satisfy

$$P^2|\psi\rangle = P|\psi\rangle \tag{3}$$

Let us apply $\Lambda_{\pm}$ to spinors which satisfy

$$\left(\not{p}-m\right)u_{\pm}(\mathbf{p}) = 0 \tag{4}$$

$$\left(\not{p}+m\right)v_{\pm}(\mathbf{p}) = 0 \tag{5}$$

Applying it to u_+ we have

$$\begin{aligned}
\Lambda_+^2 u_{\pm} &= \Lambda_+ \frac{\not{p}+m}{2m} u_{\pm} && (6)\\
&= \Lambda_+ \frac{1}{2m}\left(\not{p}-m+2m\right)u_{\pm} && (7)\\
&= \Lambda_+\left(\not{p}-m\right)u_{\pm} + u_{\pm} && (8)\\
&= \Lambda_+ u_{\pm} && (9)
\end{aligned}$$

Thus Λ_+^2 has the same effect on $u_{\pm}$ as $\Lambda_{\pm}$.

By a similar argument to above, we can find

$$\Lambda_-^2 v_\pm = -\Lambda_- \frac{\not{p} - m}{2m} v_\pm \tag{10}$$
$$= -\Lambda_- \frac{1}{2m} \left(\not{p} + m - 2m\right) v_\pm \tag{11}$$
$$= -\Lambda_- \left(\not{p} + m\right) v_\pm + v_\pm \tag{12}$$
$$= \Lambda_- v_\pm \tag{13}$$

Note that it is *not* true that $\Lambda_\pm^2 = \Lambda_\pm$ unless we apply both operators to a given state.

We can see directly that

$$\Lambda_+ v_\pm = 0 \tag{14}$$
$$\Lambda_- u_\pm = 0 \tag{15}$$

A couple of other properties are

$$\Lambda_+\Lambda_- = -\frac{1}{4m^2}\left(\not{p} + m\right)\left(\not{p} - m\right) \tag{16}$$

When applied to either $u_\pm$ or $v_\pm$ this operator gives 0 because of 4 and 5 (since the two factors on the RHS commute). Similarly $\Lambda_-\Lambda_+$ gives zero. Again, it's not correct to say the $\Lambda_+\Lambda_- = 0$ directly, since this is true only when operating on a spinor.

Finally

$$\Lambda_+ + \Lambda_- = \frac{\not{p} + m}{2m} + \frac{-\not{p} + m}{2m} = 1 \tag{17}$$

Using the same calculation as 6, we find

$$\Lambda_+ u_\pm = \frac{\not{p} + m}{2m} u_\pm \tag{18}$$
$$= u_\pm \tag{19}$$
$$\Lambda_- v_\pm = \frac{-\not{p} + m}{2m} v_\pm \tag{20}$$
$$= v_\pm \tag{21}$$

Thus $u_\pm$ and $v_\pm$ are eigenstates of $\Lambda_\pm$. Taking the Dirac conjugate gives the alternative forms

$$\overline{u}_{\pm}\Lambda_{+} = \overline{u}_{\pm} \tag{22}$$

$$\overline{v}_{\pm}\Lambda_{-} = \overline{v}_{\pm} \tag{23}$$

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