DIRAC EQUATION: NONUNIQUENESS OF SOLUTIONS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4.

Using the Dirac Hamiltonian, the Dirac equation is, in terms of the gamma matrices

$$i\frac{\partial}{\partial t}\psi(x)=\gamma^{0}\left(-i\gamma\cdot\nabla+m\right)\psi(x) \tag{1}$$

Using $\left(\gamma^{0}\right)^{2}=1$, we can multiply by γ^{0} on the left to get

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)=0 \tag{2}$$

The operation of multiplying a vector by γ^{μ} and summing is quite common when analyzing the Dirac equation, so a special notation called the *slash notation* is defined as a shorthand. This is

$$\not{a}\equiv\gamma^{\mu}a_{\mu}=\gamma_{\mu}a^{\mu} \tag{3}$$

Using this definition, the Dirac equation takes on the compact form

$$\left(i\not{\partial}-m\right)\psi(x)=0 \tag{4}$$

The gamma matrices are 4×4 matrices which satisfy the conditions

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\}=2g^{\mu\nu} \tag{5}$$

$$\left(\gamma^{\mu}\right)^{\dagger}=\gamma^{0}\gamma^{\mu}\gamma^{0} \tag{6}$$

If we define an alternative set of gamma matrices by

$$\tilde{\gamma}^{\mu}=U\gamma^{\mu}U^{\dagger} \tag{7}$$

where U is a unitary matrix, so that $U^{\dagger}=U^{-1}$, then 5 and 6 are still satisfied. For example

$$\left\{\tilde{\gamma}^{\mu},\tilde{\gamma}^{\nu}\right\} = U\gamma^{\mu}U^{\dagger}U\gamma^{\nu}U^{\dagger}+U\gamma^{\nu}U^{\dagger}U\gamma^{\mu}U^{\dagger} \tag{8}$$
$$= U\gamma^{\mu}\gamma^{\nu}U^{\dagger}+U\gamma^{\nu}\gamma^{\mu}U^{\dagger} \tag{9}$$
$$= U\left\{\gamma^{\mu},\gamma^{\nu}\right\}U^{\dagger} \tag{10}$$
$$= 2Ug^{\mu\nu}U^{\dagger} \tag{11}$$
$$= 2UU^{\dagger}g^{\mu\nu} \tag{12}$$
$$= 2g^{\mu\nu} \tag{13}$$

where the penultimate line follows from the fact that $g^{\mu\nu}$ commutes with any matrix since it is diagonal.

Because the gamma matrices are not unique, the solution $\psi(x)$ (a column vector with 4 elements) is not unique either. Suppose $\tilde{\psi}(x)$ satisfies 2 with γ^{μ} replaced by $\tilde{\gamma}^{\mu}$. Then

$$\left(i\tilde{\gamma}^{\mu}\partial_{\mu}-m\right)\tilde{\psi}(x) = \left(iU\gamma^{\mu}U^{\dagger}\partial_{\mu}-m\right)\tilde{\psi}(x) \tag{14}$$
$$= \left(iU\gamma^{\mu}U^{\dagger}\partial_{\mu}-UmU^{\dagger}\right)\tilde{\psi}(x) \tag{15}$$
$$= U\left(i\gamma^{\mu}\partial_{\mu}-m\right)U^{\dagger}\tilde{\psi}(x) \tag{16}$$

This result is equivalent to 2 if $\tilde{\psi}(x)=U\psi(x)$.

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