DIRAC EQUATION: GAMMA MATRICES AND γ^5

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.1.

We've seen the derivation of the Dirac equation earlier, and also the gamma matrices that form part of this equation, although in our earlier look at the gamma matrices, they were given explicit forms as 4×4 matrices. In fact, most of the properties of these matrices can be derived without assuming any explicit form for them, and this is the approach taken by Lahiri & Pal, so we'll follow that here.

In terms of the gamma matrices, the Dirac hamiltonian has the form

$$H=\gamma^0\left(\gamma\cdot\mathbf{p}+m\right) \tag{1}$$

where γ is a vector of three separate gamma matrices γ^i, $i=1,2,3$. L&P show that the gamma matrices satisfy the following properties:

$$\left\{\gamma^\mu,\gamma^\nu\right\}=2g^{\mu\nu} \tag{2}$$

$$\mathrm{Tr}\gamma^\mu=0 \tag{3}$$

In 2, $g^{\mu\nu}$ is the flat space metric tensor, with $g^{00}=+1$ and $g^{ii}=-1$ with all other entries being zero. Thus the gamma matrices all anticommute with each other, and all their traces are zero.

In addition to the four basic γ^μ, there are a couple of other matrices defined in terms of them. These are

$$\sigma^{\mu\nu}\equiv\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]=-\sigma^{\nu\mu} \tag{4}$$

$$\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{5}$$

$$=\frac{i}{4!}\varepsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho \tag{6}$$

Here, we'll prove a couple of properties of γ^5.

First, we calculate its trace. We have

$$\text{Tr}\,\gamma^5 = i\text{Tr}\,\left(\gamma^0\gamma^1\gamma^2\gamma^3\right) \tag{7}$$
$$= i\text{Tr}\,\left(\gamma^3\gamma^0\gamma^1\gamma^2\right) \tag{8}$$
$$= -i\text{Tr}\,\left(\gamma^0\gamma^3\gamma^1\gamma^2\right) \tag{9}$$
$$= (-1)^2\,i\text{Tr}\,\left(\gamma^0\gamma^1\gamma^3\gamma^2\right) \tag{10}$$
$$= (-1)^3\,i\text{Tr}\,\left(\gamma^0\gamma^1\gamma^2\gamma^3\right) \tag{11}$$
$$= -\text{Tr}\,\gamma^5 \tag{12}$$

The second line follows from the cyclic property of the trace. The remaining lines follow from 2, since we just swap γ^3 with each of γ^0, γ^1 and γ^2 in turn. Thus

$$\text{Tr}\,\gamma^5 = -\text{Tr}\,\gamma^5 = 0 \tag{13}$$

so γ^5 is also traceless.

We can also find the anticommutators as follows.

$$\left\{\gamma^\mu,\gamma^5\right\} = \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu \tag{14}$$

Consider the first term:

$$\gamma^\mu\gamma^5 = i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \tag{15}$$

To propagate γ^μ through the four γs to its right, we perform 3 anticommutes (with the three γ^νs where $\mu \neq \nu$) and one commute (with the γ^ν where $\nu = \mu$). Thus we will introduce three factors of -1 so the result is

$$\gamma^\mu\gamma^5 = i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \tag{16}$$
$$= -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \tag{17}$$
$$= -\gamma^5\gamma^\mu \tag{18}$$

Therefore

$$\left\{\gamma^\mu,\gamma^5\right\} = 0 \tag{19}$$

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