DIRAC EQUATION: SET OF INDEPENDENT MATRICES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.2.

In the Dirac equation, we introduced 5 γ matrices labelled γ^{μ} and γ^{5}. These matrices satisfied the properties

$$\gamma^{5} \equiv i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} \tag{1}$$

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \tag{2}$$

$$\left(\gamma^{0}\right)^{2} = 1 \tag{3}$$

$$\left(\gamma^{i}\right)^{2} = -1 \tag{4}$$

$$\mathrm{Tr}\;\gamma^{\mu} = 0 \tag{5}$$

$$\mathrm{Tr}\;\gamma^{5} = 0 \tag{6}$$

$$\left\{\gamma^{\mu},\gamma^{5}\right\} = 0 \tag{7}$$

$$(\gamma^{\mu})^{\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0} \tag{8}$$

We also introduced the matrices

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu},\gamma^{\nu}\right] = -\sigma^{\nu\mu} \tag{9}$$

We can form the set of 16 matrices defined as

$$\Gamma = \left\{1,\gamma^{\mu},\sigma^{\mu\nu},\gamma^{5}\gamma^{\mu},\gamma^{5}\right\} \tag{10}$$

We want to show that the product of any two Γ_{r}s is a linear combination of the existing Γ_{r}s.

The first thing to notice is that the elements of Γ as listed comprise matrices consisting of products of zero (that is, the unit matrix), one, two, three and four of the γ^{μ}. This is obvious for 1, γ^{μ}, $\sigma^{\mu\nu}$ and γ^{5}. To see that $\gamma^{5}\gamma^{\mu}$ is equivalent to the product of 3 γ^{ν}s, use the anticommutation rules to swap γ^{μ} into γ^{5} until it is adjacent to the γ^{ν} with $\nu=\mu$. We can now use 2 with

$\mu=\nu$ to see that $(\gamma^\mu)^2=\pm1$, so the product $\gamma^5\gamma^\mu$ reduces to the product of the three remaining γ^νs (times i).

In fact, we can use the same procedure to condense the product of any number of γ^μs down to a product of 4 or fewer γ^νs since, if there are more than 4 matrices in the product, some of them must be the same so by a process of swapping matrices using the anticommutation relations we can always bring equal matrices next to each other where they can be removed using $(\gamma^\mu)^2=\pm1$.

In passing, we can also note that, from 1e

$$\begin{aligned}
\left(\gamma^5\right)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 && (11)\\
&= -(-1)^3\left(\gamma^0\right)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && (12)\\
&= \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 && (13)\\
&= (-1)^2\left(\gamma^1\right)^2\gamma^2\gamma^3\gamma^2\gamma^3 && (14)\\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 && (15)\\
&= \left(\gamma^2\right)^2\left(\gamma^3\right)^2 && (16)\\
&= +1 && (17)
\end{aligned}$$

If the result after condensing contains zero, one, three or all four γ^νs, it is equal to some constant multiplied by 1, γ^μ, $\gamma^5\gamma^\mu$ or γ^5, respectively. If the product contains two γ^νs, we can see that it is a constant multiplied by a $\sigma^{\mu\nu}$, since

$$\begin{aligned}
\sigma^{\mu\nu} &= \frac{i}{2}\left[\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right] && (18)\\
&= \frac{i}{2}\left(2\gamma^\mu\gamma^\nu\right) && (19)\\
\gamma^\mu\gamma^\nu &= -i\sigma^{\mu\nu} && (20)
\end{aligned}$$

Thus the product of any two of the Γ_rs is some linear combination of the members of the original set Γ.

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