DIRAC EQUATION: LINEAR INDEPENDENCE OF MATRICES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.3.

In the Dirac equation, we introduced five γ matrices labelled γ^{μ} and γ^{5}. These matrices satisfied the properties

$$\gamma^{5} \equiv i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} \tag{1}$$
$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \tag{2}$$
$$\left(\gamma^{0}\right)^{2} = +1 \tag{3}$$
$$\left(\gamma^{i}\right)^{2} = -1 \tag{4}$$
$$\left(\gamma^{5}\right)^{2} = +1 \tag{5}$$
$$\mathrm{Tr}\,\gamma^{\mu} = 0 \tag{6}$$
$$\mathrm{Tr}\,\gamma^{5} = 0 \tag{7}$$
$$\left\{\gamma^{\mu},\gamma^{5}\right\} = 0 \tag{8}$$

We also introduced the matrices

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu},\gamma^{\nu}\right] = -\sigma^{\nu\mu} \tag{9}$$

We defined the set of 16 matrices defined as

$$\Gamma = \left\{1,\gamma^{\mu},\sigma^{\mu\nu},\gamma^{5}\gamma^{\mu},\gamma^{5}\right\} \tag{10}$$

and showed that the product of any number of γ^{μ}s can be reduced to a linear combination of matrices Γ_{r} from this set. We'd now like to show that the matrices in this set are linearly independent, that is, if we form the linear combination

$$\sum_{r=1}^{16} a_{r}\Gamma_{r} = 0 \tag{11}$$

then all coefficients a_r must be zero. The method of doing this is to multiply 11 successively by each of the Γ_r and then take the trace. In order to use this method, we need the traces of each of the Γ_r. We already know from 6 and 7 that the traces of all single γ^μs and γ^5 are zero. Since we're dealing with 4×4 matrices, the trace of the unit matrix 1 is 4.

Using the cyclic property of the trace, we have

$$\mathrm{Tr}\ \sigma^{\mu\nu} = \mathrm{Tr}\ (\gamma^\mu\gamma^\nu) - \mathrm{Tr}\ (\gamma^\nu\gamma^\mu) \tag{12}$$
$$= \mathrm{Tr}\ (\gamma^\mu\gamma^\nu) - \mathrm{Tr}\ (\gamma^\mu\gamma^\nu) \tag{13}$$
$$= 0 \tag{14}$$

Combining the cyclic property with 8, we see that

$$\mathrm{Tr}\ \left(\gamma^5\gamma^\mu\right) = \mathrm{Tr}\ \left(\gamma^\mu\gamma^5\right) \tag{15}$$
$$= -\mathrm{Tr}\ \left(\gamma^\mu\gamma^5\right) \tag{16}$$
$$= 0 \tag{17}$$

Therefore, the traces of all the Γ_r except the unit matrix are zero.

Now we need to consider the squares of each of the Γ_r. We saw earlier that

$$\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu \tag{18}$$

Therefore, if $\mu \neq \nu$

$$(\sigma^{\mu\nu})^2 = -\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu \tag{19}$$
$$= (\gamma^\mu)^2 (\gamma^\nu)^2 \tag{20}$$
$$= \pm 1 \tag{21}$$

where the sign depends on precise values of μ and ν, using 3 and 4.

Also, using 8 and 5

$$\left(\gamma^5\gamma^\mu\right)^2 = \gamma^5\gamma^\mu\gamma^5\gamma^\mu \tag{22}$$
$$= -\left(\gamma^5\right)^2 (\gamma^\mu)^2 \tag{23}$$
$$= \pm 1 \tag{24}$$

Thus combining these results with 3, 4 and 5, the square of every Γ_r is ± 1, that is, a multiple of the unit matrix.

Now let's return to the original statement 11 that we're trying to prove. Choose one of the members, say Γ_s, of the set, and multiply the sum by Γ_s so that we get a new sum:

$$\Gamma_s \sum_{r=1}^{16} a_r \Gamma_r = \sum_{r=1}^{16} a_r \left(\Gamma_s \Gamma_r\right) = 0 \tag{25}$$

Every term in the resulting product consists of a product of two of the Γ_r, and we know that any product of two Γ_r can be written as a linear combination of the original Γ_r. One of these terms will consist of Γ_s^2, which we've seen is a multiple of the unit matrix. The other terms will consist of other members of the set Γ, all of which have zero trace. If we take the trace of the sum 25, this is the sum of the traces of each term in the sum, and the only term that has a non-zero trace is $a_s \Gamma_s^2$, so the coefficient of this term must be $a_s = 0$ to make the total trace equal to zero. This argument applies to each of the Γ_r in turn, so in order for 25, and hence 11, to be true is if all the $a_r = 0$. QED.

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