

## DIRAC EQUATION: LORENTZ COVARIANCE

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.4.

Because the Dirac equation was derived by using the relativistic form for the energy of a particle, it should be Lorentz invariant. That is, if we look at the equation in an inertial frame moving with a constant velocity with respect to the original frame, the equation should have the same form. In their section 4.2, L&P derive the condition this invariance imposes on the spinor  $\psi$ .

The original Dirac equation is

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0 \quad (1)$$

and the transformed equation is

$$(i\gamma^\mu \partial'_\mu - m) \psi'(x') = 0 \quad (2)$$

For an infinitesimal Lorentz transformation, the transformation matrix has the form

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu} \quad (3)$$

where  $\omega_{\mu\nu} = -\omega_{\nu\mu}$  (the antisymmetry is required for the invariant length between two events) are infinitesimal quantities. The relation between  $\psi'(x')$  and  $\psi(x)$  is assumed to be linear in the quantities  $\omega_{\mu\nu}$ , so we have

$$\psi'(x') = S(\Lambda) \psi(x) \quad (4)$$

where  $S(\Lambda)$  is some matrix depending on the specific Lorentz transformation. L&P show that, for infinitesimal transformations, we must have

$$S(\Lambda) = 1 - \frac{i}{4} \beta_{\mu\nu} \omega^{\mu\nu} \quad (5)$$

where the  $4 \times 4$  matrices  $\beta_{\mu\nu}$  satisfy these commutation relations with the  $\gamma_\mu$ :

$$[\gamma_\mu, \beta_{\lambda\rho}] = 2i (g_{\mu\lambda} \gamma_\rho - g_{\mu\rho} \gamma_\lambda) \quad (6)$$

We can now show that this relation is satisfied if we choose

$$\beta_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \quad (7)$$

To verify this, we can use the anticommutators of the gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (8)$$

We need to work out terms of the form  $[\gamma_\mu, \gamma_\lambda \gamma_\rho]$ , for which we can use the identity (which can be verified by direct calculation):

$$[A, BC] = \{A, B\}C - B\{A, C\} \quad (9)$$

We then have

$$[\gamma_\mu, \sigma_{\lambda\rho}] = \frac{i}{2} [\gamma_\mu, \gamma_\lambda \gamma_\rho] - \frac{i}{2} [\gamma_\mu, \gamma_\rho \gamma_\lambda] \quad (10)$$

$$= \frac{i}{2} (\{\gamma_\mu, \gamma_\lambda\} \gamma_\rho - \gamma_\lambda \{\gamma_\mu, \gamma_\rho\} - \quad (11)$$

$$\{\gamma_\mu, \gamma_\rho\} \gamma_\lambda + \gamma_\rho \{\gamma_\mu, \gamma_\lambda\}) \quad (12)$$

$$= \frac{i}{2} 2 (g_{\mu\lambda} \gamma_\rho - g_{\mu\rho} \gamma_\lambda - g_{\mu\rho} \gamma_\lambda + g_{\mu\lambda} \gamma_\rho) \quad (13)$$

$$= 2i (g_{\mu\lambda} \gamma_\rho - g_{\mu\rho} \gamma_\lambda) \quad (14)$$

which agrees with 6.

The result for a finite Lorentz transformation is given by exponentiating 5:

$$\psi'(x') = \exp\left(-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right) \psi(x) \quad (15)$$

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The metric  $g_{\mu\nu}$  commutes with all matrices, as it is diagonal.