DIRAC EQUATION: LORENTZ COVARIANCE

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.4.

Because the Dirac equation was derived by using the relativistic form for the energy of a particle, it should be Lorentz invariant. That is, if we look at the equation in an inertial frame moving with a constant velocity with respect to the original frame, the equation should have the same form. In their section 4.2, L&P derive the condition this invariance imposes on the spinor ψ.

The original Dirac equation is

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi\left(x\right)=0 \tag{1}$$

and the transformed equation is

$$\left(i\gamma^{\mu}\partial_{\mu}^{\prime}-m\right)\psi^{\prime}\left(x^{\prime}\right)=0 \tag{2}$$

For an infinitesimal Lorentz transformation, the transformation matrix has the form

$$\Lambda_{\mu\nu}=g_{\mu\nu}+\omega_{\mu\nu} \tag{3}$$

where $\omega_{\mu\nu}=-\omega_{\nu\mu}$ (the antisymmetry is required for the invariant length between two events) are infinitesimal quantities. The relation between $\psi^{\prime}\left(x^{\prime}\right)$ and $\psi\left(x\right)$ is assumed to be linear in the quantities $\omega_{\mu\nu}$, so we have

$$\psi^{\prime}\left(x^{\prime}\right)=S\left(\Lambda\right)\psi\left(x\right) \tag{4}$$

where $S\left(\Lambda\right)$ is some matrix depending on the specific Lorentz transformation. L&P show that, for infinitesimal transformations, we must have

$$S\left(\Lambda\right)=1-\frac{i}{4}\beta_{\mu\nu}\omega^{\mu\nu} \tag{5}$$

where the 4×4 matrices $\beta_{\mu\nu}$ satisfy these commutation relations with the γ_{μ}:

$$\left[\gamma_{\mu},\beta_{\lambda\rho}\right]=2i\left(g_{\mu\lambda}\gamma_{\rho}-g_{\mu\rho}\gamma_{\lambda}\right) \tag{6}$$

We can now show that this relation is satisfied if we choose

$$\beta_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu,\gamma_\nu\right] \tag{7}$$

To verify this, we can use the anticommutators of the gamma matrices

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \tag{8}$$

We need to work out terms of the form $\left[\gamma_\mu,\gamma_\lambda\gamma_\rho\right]$, for which we can use the identity (which can be verified by direct calculation):

$$[A,BC] = \{A,B\}C - B\{A,C\} \tag{9}$$

We then have

The metric $g_{\mu\nu}$ commutes with all matrices, as it is diagonal.

$$\begin{aligned}
\left[\gamma_\mu,\sigma_{\lambda\rho}\right] &= \frac{i}{2}\left[\gamma_\mu,\gamma_\lambda\gamma_\rho\right] - \frac{i}{2}\left[\gamma_\mu,\gamma_\rho\gamma_\lambda\right] && (10)\\
&= \frac{i}{2}\left(\left\{\gamma_\mu,\gamma_\lambda\right\}\gamma_\rho - \gamma_\lambda\left\{\gamma_\mu,\gamma_\rho\right\} - \right. && (11)\\
&\left.\left\{\gamma_\mu,\gamma_\rho\right\}\gamma_\lambda + \gamma_\rho\left\{\gamma_\mu,\gamma_\lambda\right\}\right) && (12)\\
&= \frac{i}{2}2\left(g_{\mu\lambda}\gamma_\rho - g_{\mu\rho}\gamma_\lambda - g_{\mu\rho}\gamma_\lambda + g_{\mu\lambda}\gamma_\rho\right) && (13)\\
&= 2i\left(g_{\mu\lambda}\gamma_\rho - g_{\mu\rho}\gamma_\lambda\right) && (14)
\end{aligned}$$

which agrees with 6.

The result for a finite Lorentz transformation is given by exponentiating 5:

$$\psi'\left(x'\right) = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{15}$$

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