

GAMMA MATRICES: OBJECTS BEHAVING LIKE VECTORS OR TENSORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.5.

The gamma matrices that appear in the Dirac equation do not, on their own, make up a four-vector, despite the notation γ^μ . However, objects that combine the gamma matrices with the spinor ψ that is the solution of the Dirac equation *can* behave like vectors or tensors under a Lorentz transformation. Here we'll look at two such objects.

First, we look at $\bar{\psi}\gamma^\mu\psi$ where

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \quad (1)$$

We'll consider an infinitesimal Lorentz transformation given by

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu} \quad (2)$$

under which the spinor transforms according to

$$\psi'(x') = \left(1 - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right) \psi(x) \quad (3)$$

where

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \quad (4)$$

The Hermitian conjugate ψ^\dagger therefore transforms as

$$\left(\psi^\dagger(x)\right)' = \psi^\dagger(x) \left(1 + \frac{i}{4} \sigma_{\mu\nu}^\dagger \omega^{\mu\nu}\right) \quad (5)$$

We need the Hermitian conjugate of $\sigma_{\mu\nu}$, which we can find using the identities

$$\gamma_\mu^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (6)$$

$$(\gamma^0)^2 = 1 \quad (7)$$

The Hermitian conjugate of a product is the product, in reverse order, of the Hermitian conjugates.

We have

$$\sigma_{\mu\nu}^\dagger = -\frac{i}{2} \left(\gamma_\nu^\dagger \gamma_\mu^\dagger - \gamma_\mu^\dagger \gamma_\nu^\dagger \right) \quad (8)$$

$$= \frac{i}{2} \left(\gamma_\mu^\dagger \gamma_\nu^\dagger - \gamma_\nu^\dagger \gamma_\mu^\dagger \right) \quad (9)$$

$$= \frac{i}{2} \left(\gamma^0 \gamma^\mu \gamma^0 \gamma^0 \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^0 \gamma^0 \gamma^\mu \gamma^0 \right) \quad (10)$$

$$= \frac{i}{2} \left(\gamma^0 \gamma^\mu \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^\mu \gamma^0 \right) \quad (11)$$

$$= \gamma^0 \sigma_{\mu\nu} \gamma^0 \quad (12)$$

Using this result and combining 3 and 5 we get, to first order in $\omega^{\mu\nu}$:

$$(\bar{\psi} \gamma^\rho \psi)' = \psi^\dagger(x) \left(1 + \frac{i}{4} \sigma_{\mu\nu}^\dagger \omega^{\mu\nu} \right) \gamma^0 \gamma^\rho \left(1 - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \right) \psi(x) \quad (13)$$

$$= \bar{\psi} \gamma^\rho \psi + \frac{i}{4} \psi^\dagger(x) \gamma^0 \sigma_{\mu\nu} (\gamma^0)^2 \gamma^\rho \omega^{\mu\nu} \psi(x) - \quad (14)$$

$$\frac{i}{4} \psi^\dagger(x) \gamma^0 \gamma^\rho \sigma_{\mu\nu} \omega^{\mu\nu} \psi(x) \quad (15)$$

$$= \bar{\psi} \gamma^\rho \psi + \frac{i}{4} \psi^\dagger(x) \gamma^0 \sigma_{\mu\nu} \gamma^\rho \omega^{\mu\nu} \psi(x) - \quad (16)$$

$$\frac{i}{4} \psi^\dagger(x) \gamma^0 \gamma^\rho \sigma_{\mu\nu} \omega^{\mu\nu} \psi(x) \quad (17)$$

We can now use the commutator

$$[\gamma_\mu, \sigma_{\lambda\rho}] = 2i (g_{\mu\lambda} \gamma_\rho - g_{\mu\rho} \gamma_\lambda) \quad (18)$$

We can replace the terms $\gamma^\rho \sigma_{\mu\nu}$ in 17 with (after raising the index on the γ_μ in 18):

$$\gamma^\rho \sigma_{\mu\nu} = \gamma^\rho \sigma_{\mu\nu} + 2i (\delta_\mu^\rho \gamma_\nu - \delta_\nu^\rho \gamma_\mu) \quad (19)$$

With this, 17 becomes

$$(\bar{\psi} \gamma^\rho \psi)' = \bar{\psi} \gamma^\rho \psi - \frac{i}{4} \psi^\dagger(x) \gamma^0 2i (\delta_\mu^\rho \gamma_\nu - \delta_\nu^\rho \gamma_\mu) \omega^{\mu\nu} \psi(x) \quad (20)$$

$$= \bar{\psi} \gamma^\rho \psi + \frac{1}{2} \psi^\dagger(x) \gamma^0 (\gamma_\nu \omega^{\rho\nu} - \gamma_\mu \omega^{\mu\rho}) \psi(x) \quad (21)$$

Since μ and ν are dummy indices in the last term, and since $\omega^{\rho\mu} = -\omega^{\mu\rho}$, we can relabel ν as μ to get

$$(\bar{\psi}\gamma^\rho\psi)' = \bar{\psi}\gamma^\rho\psi + \frac{1}{2}\psi^\dagger(x)\gamma^0(\gamma_\mu\omega^{\rho\mu} + \gamma_\mu\omega^{\rho\mu})\psi(x) \quad (22)$$

$$= \bar{\psi}\gamma^\rho\psi + \psi^\dagger(x)\gamma^0\gamma_\mu\omega^{\rho\mu}\psi(x) \quad (23)$$

$$= \bar{\psi}\gamma^\rho\psi + \psi^\dagger(x)\gamma^0\gamma^\mu\omega_\mu^\rho\psi(x) \quad (24)$$

$$= \bar{\psi}\gamma^\rho\psi + \omega^\rho_\mu\bar{\psi}\gamma^\mu\psi \quad (25)$$

$$= (\delta^\rho_\mu + \omega^\rho_\mu)\bar{\psi}\gamma^\mu\psi \quad (26)$$

$$= \Lambda^\rho_\mu\bar{\psi}\gamma^\mu\psi \quad (27)$$

Thus $\bar{\psi}\gamma^\mu\psi$ transforms like a vector.

We can also show that $\bar{\psi}\sigma^{\mu\nu}\psi$ transforms as a second rank tensor, although it's a bit messier to do so. We start the same way:

$$(\bar{\psi}\sigma^{\alpha\beta}\psi)' = \psi^\dagger(x)\left(1 + \frac{i}{4}\sigma_{\mu\nu}^\dagger\omega^{\mu\nu}\right)\gamma^0\sigma^{\alpha\beta}\left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \quad (28)$$

$$= \bar{\psi}\sigma^{\mu\nu}\psi + \frac{i}{4}\psi^\dagger(x)\gamma^0\sigma_{\mu\nu}(\gamma^0)^2\sigma^{\alpha\beta}\omega^{\mu\nu}\psi(x) - \quad (29)$$

$$\frac{i}{4}\psi^\dagger(x)\gamma^0\sigma^{\alpha\beta}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) \quad (30)$$

$$= \bar{\psi}\sigma^{\alpha\beta}\psi + \frac{i}{4}\psi^\dagger(x)\gamma^0\sigma_{\mu\nu}\sigma^{\alpha\beta}\omega^{\mu\nu}\psi(x) - \quad (31)$$

$$\frac{i}{4}\psi^\dagger(x)\gamma^0\sigma^{\alpha\beta}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) \quad (32)$$

Now we need the commutator $[\sigma^{\alpha\beta}, \sigma_{\mu\nu}]$. Since

$$\sigma^{\alpha\beta} = \frac{i}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) \quad (33)$$

we can work out the first term in the commutator using 18 (raising indices where appropriate):

$$[\gamma^\alpha\gamma^\beta, \sigma_{\mu\nu}] = \gamma^\alpha\gamma^\beta\sigma_{\mu\nu} - \sigma_{\mu\nu}\gamma^\alpha\gamma^\beta \quad (34)$$

$$= \gamma^\alpha\left(\sigma_{\mu\nu}\gamma^\beta + 2i\left(\delta_\mu^\beta\gamma_\nu - \delta_\nu^\beta\gamma_\mu\right)\right) - \sigma_{\mu\nu}\gamma^\alpha\gamma^\beta \quad (35)$$

$$= \sigma_{\mu\nu}\gamma^\alpha\gamma^\beta + 2i\left(\delta_\mu^\alpha\gamma_\nu\gamma^\beta - \delta_\nu^\alpha\gamma_\mu\gamma^\beta\right) + \quad (36)$$

$$2i\left(\delta_\mu^\beta\gamma^\alpha\gamma_\nu - \delta_\nu^\beta\gamma^\alpha\gamma_\mu\right) - \sigma_{\mu\nu}\gamma^\alpha\gamma^\beta \quad (37)$$

$$= 2i\left(\delta_\mu^\alpha\gamma_\nu\gamma^\beta - \delta_\nu^\alpha\gamma_\mu\gamma^\beta + \delta_\mu^\beta\gamma^\alpha\gamma_\nu - \delta_\nu^\beta\gamma^\alpha\gamma_\mu\right) \quad (38)$$

The commutator of the second term in 33 is the same but with $\alpha \leftrightarrow \beta$:

$$\left[\gamma^\alpha \gamma^\beta, \sigma_{\mu\nu} \right] = 2i \left(\delta_\mu^\beta \gamma_\nu \gamma^\alpha - \delta_\nu^\beta \gamma_\mu \gamma^\alpha + \delta_\mu^\alpha \gamma^\beta \gamma_\nu - \delta_\nu^\alpha \gamma^\beta \gamma_\mu \right) \quad (39)$$

Combining 38 and 39, we get

$$\left[\sigma^{\alpha\beta}, \sigma_{\mu\nu} \right] = \frac{i}{2} 2i \left(\delta_\mu^\alpha \left(\gamma_\nu \gamma^\beta - \gamma^\beta \gamma_\nu \right) + \delta_\mu^\beta \left(\gamma^\alpha \gamma_\nu - \gamma_\nu \gamma^\alpha \right) - \right. \quad (40)$$

$$\left. \delta_\nu^\alpha \left(\gamma_\mu \gamma^\beta - \gamma^\beta \gamma_\mu \right) - \delta_\nu^\beta \left(\gamma^\alpha \gamma_\mu - \gamma_\mu \gamma^\alpha \right) \right) \quad (41)$$

$$= \frac{2}{i} \left(\delta_\mu^\alpha \sigma^\beta{}_\nu - \delta_\mu^\beta \sigma^\alpha{}_\nu - \delta_\nu^\alpha \sigma^\beta{}_\mu + \delta_\nu^\beta \sigma^\alpha{}_\mu \right) \quad (42)$$

Using this in 32 we have

$$\left(\bar{\psi} \sigma^{\alpha\beta} \psi \right)' = \bar{\psi} \sigma^{\alpha\beta} \psi + \frac{i}{4} \psi^\dagger(x) \gamma^0 \omega^{\mu\nu} \times \quad (43)$$

$$\frac{2}{i} \left(-\delta_\mu^\alpha \sigma^\beta{}_\nu + \delta_\mu^\beta \sigma^\alpha{}_\nu + \delta_\nu^\alpha \sigma^\beta{}_\mu - \delta_\nu^\beta \sigma^\alpha{}_\mu \right) \psi(x) \quad (44)$$

$$= \bar{\psi} \sigma^{\alpha\beta} \psi + \frac{1}{2} \psi^\dagger(x) \gamma^0 \times \quad (45)$$

$$\left(-\omega^{\alpha\nu} \sigma^\beta{}_\nu + \omega^{\beta\nu} \sigma^\alpha{}_\nu + \omega^{\mu\alpha} \sigma^\beta{}_\mu - \omega^{\mu\beta} \sigma^\alpha{}_\mu \right) \psi(x) \quad (46)$$

$$= \bar{\psi} \sigma^{\alpha\beta} \psi + \psi^\dagger(x) \gamma^0 \left(-\omega^{\alpha\mu} \sigma^\beta{}_\mu + \omega^{\beta\nu} \sigma^\alpha{}_\nu \right) \psi(x) \quad (47)$$

$$= \bar{\psi} \sigma^{\alpha\beta} \psi + \psi^\dagger(x) \gamma^0 \left(-\omega^\alpha{}_\mu \sigma^{\beta\mu} + \omega^\beta{}_\nu \sigma^{\alpha\nu} \right) \psi(x) \quad (48)$$

$$\bar{\psi} \sigma^{\alpha\beta} \psi + \psi^\dagger(x) \gamma^0 \left(\omega^\alpha{}_\mu \sigma^{\mu\beta} + \omega^\beta{}_\nu \sigma^{\alpha\nu} \right) \psi(x) \quad (49)$$

where we used $\sigma^{\beta\mu} = -\sigma^{\mu\beta}$ to get the last line.

With 2 with the second index lowered, we have

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad (50)$$

and to first order in $\omega^\mu{}_\nu$ 49 is

$$\left(\bar{\psi} \sigma^{\alpha\beta} \psi \right)' = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \bar{\psi} \sigma^{\mu\nu} \psi \quad (51)$$

Thus $\bar{\psi} \sigma^{\alpha\beta} \psi$ transforms as a second-rank tensor.