GAMMA MATRICES: OBJECTS BEHAVING LIKE VECTORS OR TENSORS

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Post date: 7 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.5.

The gamma matrices that appear in the Dirac equation do not, on their own, make up a four-vector, despite the notation γ^{μ}. However, objects that combine the gamma matrices with the spinor ψ that is the solution of the Dirac equation *can* behave like vectors or tensors under a Lorentz transformation. Here we'll look at two such objects.

First, we look at $\bar{\psi}\gamma^{\mu}\psi$ where

$$\bar{\psi}\equiv\psi^{\dagger}\gamma^{0} \tag{1}$$

We'll consider an infinitesimal Lorentz transformation given by

$$\Lambda^{\mu\nu}=g^{\mu\nu}+\omega^{\mu\nu} \tag{2}$$

under which the spinor transforms according to

$$\psi'\left(x'\right)=\left(1-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi\left(x\right) \tag{3}$$

where

$$\sigma_{\mu\nu}=\frac{i}{2}\left[\gamma_{\mu},\gamma_{\nu}\right] \tag{4}$$

The Hermitian conjugate $\psi^{\dagger}$ therefore transforms as

$$\left(\psi^{\dagger}\left(x\right)\right)'=\psi^{\dagger}\left(x\right)\left(1+\frac{i}{4}\sigma_{\mu\nu}^{\dagger}\omega^{\mu\nu}\right) \tag{5}$$

The Hermitian conjugate of a product is the product, in reverse order, of the Hermitian conjugates.

We need the Hermitian conjugate of $\sigma_{\mu\nu}$, which we can find using the identities

$$\gamma_{\mu}^{\dagger}=\gamma^{0}\gamma^{\mu}\gamma^{0} \tag{6}$$

$$\left(\gamma^{0}\right)^{2}=1 \tag{7}$$

We have

$$\sigma_{\mu\nu}^{\dagger} = -\frac{i}{2}\left(\gamma_{\nu}^{\dagger}\gamma_{\mu}^{\dagger} - \gamma_{\mu}^{\dagger}\gamma_{\nu}^{\dagger}\right) \tag{8}$$

$$= \frac{i}{2}\left(\gamma_{\mu}^{\dagger}\gamma_{\nu}^{\dagger} - \gamma_{\nu}^{\dagger}\gamma_{\mu}^{\dagger}\right) \tag{9}$$

$$= \frac{i}{2}\left(\gamma^{0}\gamma^{\mu}\gamma^{0}\gamma^{0}\gamma^{\nu}\gamma^{0} - \gamma^{0}\gamma^{\nu}\gamma^{0}\gamma^{0}\gamma^{\mu}\gamma^{0}\right) \tag{10}$$

$$= \frac{i}{2}\left(\gamma^{0}\gamma^{\mu}\gamma^{\nu}\gamma^{0} - \gamma^{0}\gamma^{\nu}\gamma^{\mu}\gamma^{0}\right) \tag{11}$$

$$= \gamma^{0}\sigma_{\mu\nu}\gamma^{0} \tag{12}$$

Using this result and combining 3 and 5 we get, to first order in $\omega^{\mu\nu}$:

$$\left(\bar{\psi}\gamma^{\rho}\psi\right)' = \psi^{\dagger}(x)\left(1+\frac{i}{4}\sigma_{\mu\nu}^{\dagger}\omega^{\mu\nu}\right)\gamma^{0}\gamma^{\rho}\left(1-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{13}$$

$$= \bar{\psi}\gamma^{\rho}\psi + \frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\sigma_{\mu\nu}\left(\gamma^{0}\right)^{2}\gamma^{\rho}\omega^{\mu\nu}\psi(x) - \tag{14}$$

$$\frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\gamma^{\rho}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) \tag{15}$$

$$= \bar{\psi}\gamma^{\rho}\psi + \frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\sigma_{\mu\nu}\gamma^{\rho}\omega^{\mu\nu}\psi(x) - \tag{16}$$

$$\frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\gamma^{\rho}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) \tag{17}$$

We can now use the commutator

$$\left[\gamma_{\mu},\sigma_{\lambda\rho}\right] = 2i\left(g_{\mu\lambda}\gamma_{\rho} - g_{\mu\rho}\gamma_{\lambda}\right) \tag{18}$$

We can replace the terms $\gamma^{\rho}\sigma_{\mu\nu}$ in 17 with (after raising the index on the γ_{μ} in 18):

$$\gamma^{\rho}\sigma_{\mu\nu} = \gamma^{\rho}\sigma_{\mu\nu} + 2i\left(\delta_{\mu}^{\rho}\gamma_{\nu} - \delta_{\nu}^{\rho}\gamma_{\mu}\right) \tag{19}$$

With this, 17 becomes

$$\left(\bar{\psi}\gamma^{\rho}\psi\right)' = \bar{\psi}\gamma^{\rho}\psi - \frac{i}{4}\psi^{\dagger}(x)\gamma^{0}2i\left(\delta_{\mu}^{\rho}\gamma_{\nu} - \delta_{\nu}^{\rho}\gamma_{\mu}\right)\omega^{\mu\nu}\psi(x) \tag{20}$$

$$= \bar{\psi}\gamma^{\rho}\psi + \frac{1}{2}\psi^{\dagger}(x)\gamma^{0}\left(\gamma_{\nu}\omega^{\rho\nu} - \gamma_{\mu}\omega^{\mu\rho}\right)\psi(x) \tag{21}$$

Since μ and ν are dummy indices in the last term, and since $\omega^{\rho\mu} = -\omega^{\mu\rho}$, we can relabel ν as μ to get

$$\left(\bar{\psi}\gamma^{\rho}\psi\right)' = \bar{\psi}\gamma^{\rho}\psi + \frac{1}{2}\psi^{\dagger}(x)\gamma^{0}\left(\gamma_{\mu}\omega^{\rho\mu} + \gamma_{\mu}\omega^{\rho\mu}\right)\psi(x) \tag{22}$$

$$= \bar{\psi}\gamma^{\rho}\psi + \psi^{\dagger}(x)\gamma^{0}\gamma_{\mu}\omega^{\rho\mu}\psi(x) \tag{23}$$

$$= \bar{\psi}\gamma^{\rho}\psi + \psi^{\dagger}(x)\gamma^{0}\gamma^{\mu}\omega^{\rho}{}_{\mu}\psi(x) \tag{24}$$

$$= \bar{\psi}\gamma^{\rho}\psi + \omega^{\rho}{}_{\mu}\bar{\psi}\gamma^{\mu}\psi \tag{25}$$

$$= \left(\delta^{\rho}_{\mu} + \omega^{\rho}{}_{\mu}\right)\bar{\psi}\gamma^{\mu}\psi \tag{26}$$

$$= \Lambda^{\rho}{}_{\mu}\bar{\psi}\gamma^{\mu}\psi \tag{27}$$

Thus $\bar{\psi}\gamma^{\mu}\psi$ transforms like a vector.

We can also show that $\bar{\psi}\sigma^{\mu\nu}\psi$ transforms as a second rank tensor, although it's a bit messier to do so. We start the same way:

$$\left(\bar{\psi}\sigma^{\alpha\beta}\psi\right)' = \psi^{\dagger}(x)\left(1 + \frac{i}{4}\sigma^{\dagger}_{\mu\nu}\omega^{\mu\nu}\right)\gamma^{0}\sigma^{\alpha\beta}\left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{28}$$

$$= \bar{\psi}\sigma^{\mu\nu}\psi + \frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\sigma_{\mu\nu}\left(\gamma^{0}\right)^{2}\sigma^{\alpha\beta}\omega^{\mu\nu}\psi(x) - \tag{29}$$

$$\frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\sigma^{\alpha\beta}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) \tag{30}$$

$$= \bar{\psi}\sigma^{\alpha\beta}\psi + \frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\sigma_{\mu\nu}\sigma^{\alpha\beta}\omega^{\mu\nu}\psi(x) - \tag{31}$$

$$\frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\sigma^{\alpha\beta}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) \tag{32}$$

Now we need the commutator $\left[\sigma^{\alpha\beta}, \sigma_{\mu\nu}\right]$. Since

$$\sigma^{\alpha\beta} = \frac{i}{2}\left(\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha}\right) \tag{33}$$

we can work out the first term in the commutator using 18 (raising indices where appropriate):

$$\left[\gamma^{\alpha}\gamma^{\beta},\sigma_{\mu\nu}\right]=\gamma^{\alpha}\gamma^{\beta}\sigma_{\mu\nu}-\sigma_{\mu\nu}\gamma^{\alpha}\gamma^{\beta} \tag{34}$$

$$=\gamma^{\alpha}\left(\sigma_{\mu\nu}\gamma^{\beta}+2i\left(\delta_{\mu}^{\beta}\gamma_{\nu}-\delta_{\nu}^{\beta}\gamma_{\mu}\right)\right)-\sigma_{\mu\nu}\gamma^{\alpha}\gamma^{\beta} \tag{35}$$

$$=\sigma_{\mu\nu}\gamma^{\alpha}\gamma^{\beta}+2i\left(\delta_{\mu}^{\alpha}\gamma_{\nu}\gamma^{\beta}-\delta_{\nu}^{\alpha}\gamma_{\mu}\gamma^{\beta}\right)+ \tag{36}$$

$$2i\left(\delta_{\mu}^{\beta}\gamma^{\alpha}\gamma_{\nu}-\delta_{\nu}^{\beta}\gamma^{\alpha}\gamma_{\mu}\right)-\sigma_{\mu\nu}\gamma^{\alpha}\gamma^{\beta} \tag{37}$$

$$=2i\left(\delta_{\mu}^{\alpha}\gamma_{\nu}\gamma^{\beta}-\delta_{\nu}^{\alpha}\gamma_{\mu}\gamma^{\beta}+\delta_{\mu}^{\beta}\gamma^{\alpha}\gamma_{\nu}-\delta_{\nu}^{\beta}\gamma^{\alpha}\gamma_{\mu}\right) \tag{38}$$

The commutator of the second term in 33 is the same but with $\alpha\leftrightarrow\beta$:

$$\left[\gamma^{\alpha}\gamma^{\beta},\sigma_{\mu\nu}\right]=2i\left(\delta_{\mu}^{\beta}\gamma_{\nu}\gamma^{\alpha}-\delta_{\nu}^{\beta}\gamma_{\mu}\gamma^{\alpha}+\delta_{\mu}^{\alpha}\gamma^{\beta}\gamma_{\nu}-\delta_{\nu}^{\alpha}\gamma^{\beta}\gamma_{\mu}\right) \tag{39}$$

Combining 38 and 39, we get

$$\left[\sigma^{\alpha\beta},\sigma_{\mu\nu}\right]=\frac{i}{2}2i\left(\delta_{\mu}^{\alpha}\left(\gamma_{\nu}\gamma^{\beta}-\gamma^{\beta}\gamma_{\nu}\right)+\delta_{\mu}^{\beta}\left(\gamma^{\alpha}\gamma_{\nu}-\gamma_{\nu}\gamma^{\alpha}\right)-\right. \tag{40}$$

$$\left.\delta_{\nu}^{\alpha}\left(\gamma_{\mu}\gamma^{\beta}-\gamma^{\beta}\gamma_{\mu}\right)-\delta_{\nu}^{\beta}\left(\gamma^{\alpha}\gamma_{\mu}-\gamma_{\mu}\gamma^{\alpha}\right)\right) \tag{41}$$

$$=\frac{2}{i}\left(\delta_{\mu}^{\alpha}\sigma^{\beta}{}_{\nu}-\delta_{\mu}^{\beta}\sigma^{\alpha}{}_{\nu}-\delta_{\nu}^{\alpha}\sigma^{\beta}{}_{\mu}+\delta_{\nu}^{\beta}\sigma^{\alpha}{}_{\mu}\right) \tag{42}$$

Using this in 32 we have

$$\left(\bar{\psi}\sigma^{\alpha\beta}\psi\right)'=\bar{\psi}\sigma^{\alpha\beta}\psi+\frac{i}{4}\psi^{\dagger}(x)\gamma^{0}\omega^{\mu\nu}\times \tag{43}$$

$$\frac{2}{i}\left(-\delta_{\mu}^{\alpha}\sigma^{\beta}{}_{\nu}+\delta_{\mu}^{\beta}\sigma^{\alpha}{}_{\nu}+\delta_{\nu}^{\alpha}\sigma^{\beta}{}_{\mu}-\delta_{\nu}^{\beta}\sigma^{\alpha}{}_{\mu}\right)\psi(x) \tag{44}$$

$$=\bar{\psi}\sigma^{\alpha\beta}\psi+\frac{1}{2}\psi^{\dagger}(x)\gamma^{0}\times \tag{45}$$

$$\left(-\omega^{\alpha\nu}\sigma^{\beta}{}_{\nu}+\omega^{\beta\nu}\sigma^{\alpha}{}_{\nu}+\omega^{\mu\alpha}\sigma^{\beta}{}_{\mu}-\omega^{\mu\beta}\sigma^{\alpha}{}_{\mu}\right)\psi(x) \tag{46}$$

$$=\bar{\psi}\sigma^{\alpha\beta}\psi+\psi^{\dagger}(x)\gamma^{0}\left(-\omega^{\alpha\mu}\sigma^{\beta}{}_{\mu}+\omega^{\beta\nu}\sigma^{\alpha}{}_{\nu}\right)\psi(x) \tag{47}$$

$$=\bar{\psi}\sigma^{\alpha\beta}\psi+\psi^{\dagger}(x)\gamma^{0}\left(-\omega^{\alpha}{}_{\mu}\sigma^{\beta\mu}+\omega^{\beta}{}_{\nu}\sigma^{\alpha\nu}\right)\psi(x) \tag{48}$$

$$\bar{\psi}\sigma^{\alpha\beta}\psi+\psi^{\dagger}(x)\gamma^{0}\left(\omega^{\alpha}{}_{\mu}\sigma^{\mu\beta}+\omega^{\beta}{}_{\nu}\sigma^{\alpha\nu}\right)\psi(x) \tag{49}$$

where we used $\sigma^{\beta\mu}=-\sigma^{\mu\beta}$ to get the last line.

With 2 with the second index lowered, we have

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \tag{50}$$

and to first order in $\omega^{\mu}{}_{\nu}$ 49 is

$$\left(\bar{\psi}\sigma^{\alpha\beta}\psi\right)' = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\bar{\psi}\sigma^{\mu\nu}\psi \tag{51}$$

Thus $\bar{\psi}\sigma^{\alpha\beta}\psi$ transforms as a second-rank tensor.