# GAMMA MATRICES: A CURIOUS EXAMPLE

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.6.

Here is an unusual example of an object that transforms the same way as a Dirac spinor $\psi(x)$ under a Lorentz transformation.

Suppose there is a matrix $C$ such that

$$C^{-1}\gamma_{\mu}C=-\gamma_{\mu}^{\mathsf{T}} \tag{1}$$

where the superscript $\mathsf{T}$ indicates a matrix transpose. We want to show that the object

$$\psi_{C}\equiv C\gamma_{0}^{\mathsf{T}}\psi^{*} \tag{2}$$

where a star indicates complex conjugate, transforms in the same way as $\psi$ itself.

The key to this problem lies in the fact that since we have no information on how the transpose of a gamma matrix behaves, we need to get rid of transposes in the equations. From 1, we see that

$$C\gamma_{\mu}^{\mathsf{T}}=-\gamma_{\mu}C \tag{3}$$

so this offers a way of getting rid of transposes.

The spinor $\psi$ transforms according to (for an infinitesimal transformation):

$$\psi^{\prime}\left(x^{\prime}\right)=\left(1-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{4}$$

where

$$\sigma_{\mu\nu}=\frac{i}{2}\left[\gamma_{\mu},\gamma_{\nu}\right] \tag{5}$$

The complex conjugate therefore transforms as (the elements $\omega^{\mu\nu}$ of the transformation are real):

$$\psi^{*\prime}\left(x^{\prime}\right)=\left(1+\frac{i}{4}\sigma_{\mu\nu}^{*}\omega^{\mu\nu}\right)\psi^{*}\left(x\right) \tag{6}$$

$$=\left(1-\frac{i}{4}\frac{i}{2}\left(\gamma_{\mu}^{*}\gamma_{\nu}^{*}-\gamma_{\nu}^{*}\gamma_{\mu}^{*}\right)\omega^{\mu\nu}\right)\psi^{*}\left(x\right) \tag{7}$$

The hermitian conjugate of a matrix is the complex conjugate followed by the transpose. Therefore using the identity

$$\gamma_{\mu}^{\dagger}=\gamma_{0}\gamma_{\mu}\gamma_{0} \tag{8}$$

we have

$$\gamma_{\mu}^{*}=\left(\gamma_{\mu}^{\dagger}\right)^{\mathsf{T}} \tag{9}$$

$$=\gamma_{0}^{\mathsf{T}}\gamma_{\mu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}} \tag{10}$$

Inserting this into 7 we get

$$\psi^{*\prime}\left(x^{\prime}\right)=\psi^{*}\left(x\right)-\frac{i}{4}\frac{i}{2}\left(\gamma_{0}^{\mathsf{T}}\gamma_{\mu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{\nu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}-\right. \tag{11}$$

$$\left.\gamma_{0}^{\mathsf{T}}\gamma_{\nu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{\mu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\right)\omega^{\mu\nu}\psi^{*}\left(x\right) \tag{12}$$

Inserting this into 2 we get

$$\left(C\gamma_{0}^{\mathsf{T}}\psi^{*}\right)^{\prime}=C\gamma_{0}^{\mathsf{T}}\psi^{*}\left(x\right)-\frac{i}{4}\frac{i}{2}\left(C\gamma_{0}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{\mu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{\nu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}-\right. \tag{13}$$

$$\left.C\gamma_{0}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{\nu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\gamma_{\mu}^{\mathsf{T}}\gamma_{0}^{\mathsf{T}}\right)\omega^{\mu\nu}\psi^{*}\left(x\right) \tag{14}$$

We can now use the property 3 to move the matrix $C$ through the product of transposed gamma matrices. As we move $C$ one matrix to the right, we introduce a factor of $-1$ and also convert the transpose back to the original gamma matrix. If we move $C$ through the first 6 matrices in each term, we get a factor of $(-1)^{6}=+1$ and convert these 6 gamma matrix transposes back to their original forms, so we have, using $(\gamma_{0})^{2}=1$:

$$\left(C\gamma_0^{\mathsf{T}}\psi^*\right)' = C\gamma_0^{\mathsf{T}}\psi^*(x) - \frac{i}{4}\frac{i}{2}\Big(\gamma_0\gamma_0\gamma_\mu\gamma_0\gamma_0\gamma_\nu C\gamma_0^{\mathsf{T}} - \tag{15}$$

$$\gamma_0\gamma_0\gamma_\nu\gamma_0\gamma_0\gamma_\mu C\gamma_0^{\mathsf{T}}\Big)\omega^{\mu\nu}\psi^*(x) \tag{16}$$

$$= C\gamma_0^{\mathsf{T}}\psi^*(x) - \frac{i}{4}\frac{i}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right)C\gamma_0^{\mathsf{T}}\omega^{\mu\nu}\psi^*(x) \tag{17}$$

$$= C\gamma_0^{\mathsf{T}}\psi^*(x) - \frac{i}{4}\sigma_{\mu\nu}C\gamma_0^{\mathsf{T}}\omega^{\mu\nu}\psi^*(x) \tag{18}$$

$$= \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)C\gamma_0^{\mathsf{T}}\psi^*(x) \tag{19}$$

Thus $C\gamma_0^{\mathsf{T}}\psi^*(x)$ transforms the same way as $\psi(x)$ in 4.

I'm not sure if this result has any applications, but it is quite a clever exercise.