

GAMMA MATRICES: A CURIOUS EXAMPLE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 7 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.6.

Here is an unusual example of an object that transforms the same way as a Dirac spinor $\psi(x)$ under a Lorentz transformation.

Suppose there is a matrix C such that

$$C^{-1}\gamma_{\mu}C = -\gamma_{\mu}^{\text{T}} \quad (1)$$

where the superscript T indicates a matrix transpose. We want to show that the object

$$\psi_C \equiv C\gamma_0^{\text{T}}\psi^* \quad (2)$$

where a star indicates complex conjugate, transforms in the same way as ψ itself.

The key to this problem lies in the fact that since we have no information on how the transpose of a gamma matrix behaves, we need to get rid of transposes in the equations. From 1, we see that

$$C\gamma_{\mu}^{\text{T}} = -\gamma_{\mu}C \quad (3)$$

so this offers a way of getting rid of transposes.

The spinor ψ transforms according to (for an infinitesimal transformation):

$$\psi'(x') = \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \quad (4)$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}] \quad (5)$$

The complex conjugate therefore transforms as (the elements $\omega^{\mu\nu}$ of the transformation are real):

$$\psi^{*'}(x') = \left(1 + \frac{i}{4} \sigma_{\mu\nu}^* \omega^{\mu\nu}\right) \psi^*(x) \quad (6)$$

$$= \left(1 - \frac{i}{4} \frac{i}{2} (\gamma_\mu^* \gamma_\nu^* - \gamma_\nu^* \gamma_\mu^*) \omega^{\mu\nu}\right) \psi^*(x) \quad (7)$$

The hermitian conjugate of a matrix is the complex conjugate followed by the transpose. Therefore using the identity

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \quad (8)$$

we have

$$\gamma_\mu^* = (\gamma_\mu^\dagger)^\top \quad (9)$$

$$= \gamma_0^\top \gamma_\mu^\top \gamma_0^\top \quad (10)$$

Inserting this into 7 we get

$$\psi^{*'}(x') = \psi^*(x) - \frac{i}{4} \frac{i}{2} (\gamma_0^\top \gamma_\mu^\top \gamma_0^\top \gamma_0^\top \gamma_\nu^\top \gamma_0^\top - \quad (11)$$

$$\gamma_0^\top \gamma_\nu^\top \gamma_0^\top \gamma_0^\top \gamma_\mu^\top \gamma_0^\top) \omega^{\mu\nu} \psi^*(x) \quad (12)$$

Inserting this into 2 we get

$$(C \gamma_0^\top \psi^*)' = C \gamma_0^\top \psi^*(x) - \frac{i}{4} \frac{i}{2} (C \gamma_0^\top \gamma_0^\top \gamma_\mu^\top \gamma_0^\top \gamma_0^\top \gamma_\nu^\top \gamma_0^\top - \quad (13)$$

$$C \gamma_0^\top \gamma_0^\top \gamma_\nu^\top \gamma_0^\top \gamma_0^\top \gamma_\mu^\top \gamma_0^\top) \omega^{\mu\nu} \psi^*(x) \quad (14)$$

We can now use the property 3 to move the matrix C through the product of transposed gamma matrices. As we move C one matrix to the right, we introduce a factor of -1 and also convert the transpose back to the original gamma matrix. If we move C through the first 6 matrices in each term, we get a factor of $(-1)^6 = +1$ and convert these 6 gamma matrix transposes back to their original forms, so we have, using $(\gamma_0)^2 = 1$:

$$\left(C\gamma_0^\top \psi^*\right)' = C\gamma_0^\top \psi^*(x) - \frac{i}{4} \frac{i}{2} \left(\gamma_0 \gamma_0 \gamma_\mu \gamma_0 \gamma_0 \gamma_\nu C\gamma_0^\top - \right. \quad (15)$$

$$\left. \gamma_0 \gamma_0 \gamma_\nu \gamma_0 \gamma_0 \gamma_\mu C\gamma_0^\top\right) \omega^{\mu\nu} \psi^*(x) \quad (16)$$

$$= C\gamma_0^\top \psi^*(x) - \frac{i}{4} \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) C\gamma_0^\top \omega^{\mu\nu} \psi^*(x) \quad (17)$$

$$= C\gamma_0^\top \psi^*(x) - \frac{i}{4} \sigma_{\mu\nu} C\gamma_0^\top \omega^{\mu\nu} \psi^*(x) \quad (18)$$

$$= \left(1 - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right) C\gamma_0^\top \psi^*(x) \quad (19)$$

Thus $C\gamma_0^\top \psi^*(x)$ transforms the same way as $\psi(x)$ in 4.

I'm not sure if this result has any applications, but it is quite a clever exercise.