

## DIRAC SPINORS: NORMALIZATION

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Post date: 11 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.8.

The Dirac equation can be written as

$$(i\cancel{\partial} - m)\psi(x) = 0 \quad (1)$$

This is the equation for a free particle, without any potential or interaction terms. In the rest frame of the particle,  $\mathbf{p} = 0$  so this equation then contains only the term involving  $p^0$ . That is

$$(i\gamma^0\partial_0 - m)\psi(x) = 0 \quad (2)$$

With a plane wave solution in the rest frame, we have

$$\psi(x) = e^{-ip \cdot x} = e^{-iEt} \quad (3)$$

so we have

$$(\gamma^0 E - m)\psi(x) = 0 \quad (4)$$

Because of the condition

$$(\gamma^0)^2 = 1 \quad (5)$$

the eigenvalues of  $\gamma^0$  are  $\pm 1$ , so the energy  $E$  in 4 must be

$$E = \pm m \quad (6)$$

That is, the energy can be positive or negative, so the Dirac equation suffers from the same problem as the Klein-Gordon equation in that it allows negative energies.

Because the Dirac equation involves the  $4 \times 4$  gamma matrices, the solution vector  $\psi(x)$  is a 4-component vector, which we can represent as

$$\psi(x) \sim \begin{cases} u_s(\mathbf{p}) e^{-ip \cdot x} \\ v_s(\mathbf{p}) e^{ip \cdot x} \end{cases} \quad (7)$$

where  $u_s$  and  $v_s$  are 4-component column vectors (spinors) and  $s$  labels the two independent solutions of each column vector.

Putting 7 into 1 gives us equations for the spinors:

$$(\not{p} - m) u_s(\mathbf{p}) = 0 \quad (8)$$

$$(\not{p} + m) v_s(\mathbf{p}) = 0 \quad (9)$$

Taking the Dirac conjugate, with

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \quad (10)$$

we have from 8

$$(\gamma^\mu p_\mu - m) u_s(\mathbf{p}) = 0 \quad (11)$$

Taking the hermitian conjugate and using

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \quad (12)$$

gives

$$u_s^\dagger(\mathbf{p}) (\gamma^{\mu\dagger} p_\mu - m) = u_s^\dagger(\mathbf{p}) (\gamma^0 \gamma^\mu \gamma^0 p_\mu - \gamma^0 m \gamma^0) \quad (13)$$

$$= \bar{u}_s(\mathbf{p}) (\gamma^\mu \gamma^0 p_\mu - m \gamma^0) \quad (14)$$

$$= \bar{u}_s(\mathbf{p}) (\not{p} - m) \gamma^0 \quad (15)$$

$$= 0 \quad (16)$$

Multiplying on the right by  $\gamma^0$  gives

$$\bar{u}_s(\mathbf{p}) (\not{p} - m) = 0 \quad (17)$$

$$\bar{v}_s(\mathbf{p}) (\not{p} + m) = 0 \quad (18)$$

where the second equation follows from a similar calculation on 9.

At this point L&P state the normalization of the spinors to be

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = 2E_p \delta_{rs} \quad (19)$$

$$v_r^\dagger(\mathbf{p}) u_s(-\mathbf{p}) = u_r^\dagger(\mathbf{p}) v_s(-\mathbf{p}) = 0 \quad (20)$$

We can derive a couple of useful formulas. From 8 (with subscript  $r$ ) we multiply from the left by  $\bar{u}_s \gamma^\mu$  and 17 from the right by  $\gamma^\mu u_r$  (all  $u_s$  are assumed to be functions of  $\mathbf{p}$ ) and add:

$$\bar{u}_s \gamma^\mu (\gamma^\alpha p_\alpha - m) u_r + \bar{u}_s (\gamma^\alpha p_\alpha - m) \gamma^\mu u_r = \quad (21)$$

$$\bar{u}_s (\gamma^\mu \gamma^\alpha + \gamma^\alpha \gamma^\mu) p_\alpha u_r - 2m \bar{u}_s \gamma^\mu u_r = \quad (22)$$

$$2\bar{u}_s g^{\mu\alpha} p_\alpha u_r - 2m \bar{u}_s \gamma^\mu u_r = 0 \quad (23)$$

We therefore have

$$\bar{u}_s(\mathbf{p}) p^\mu u_r(\mathbf{p}) = m \bar{u}_s(\mathbf{p}) \gamma^\mu u_r(\mathbf{p}) \quad (24)$$

A similar relation for  $v_s$  follows from 9 and 18, with  $m$  replaced by  $-m$ :

$$\bar{v}_s(\mathbf{p}) p^\mu v_r(\mathbf{p}) = -m \bar{v}_s(\mathbf{p}) \gamma^\mu v_r(\mathbf{p}) \quad (25)$$

With  $\mu = 0$  we have from 24 and 19

$$m \bar{u}_s(\mathbf{p}) \gamma^0 u_r(\mathbf{p}) = m u_s^\dagger(\mathbf{p}) (\gamma^0)^2 u_r(\mathbf{p}) \quad (26)$$

$$= m u_s^\dagger(\mathbf{p}) u_r(\mathbf{p}) \quad (27)$$

$$= 2m E_p \delta_{rs} \quad (28)$$

From the LHS of 24 this is equal to

$$2m E_p \delta_{rs} = \bar{u}_s(\mathbf{p}) p^0 u_r(\mathbf{p}) \quad (29)$$

$$= E_p \bar{u}_s(\mathbf{p}) u_r(\mathbf{p}) \quad (30)$$

So we get an alternative form of the normalization conditions

$$\bar{u}_s(\mathbf{p}) u_r(\mathbf{p}) = 2m \delta_{rs} \quad (31)$$

$$\bar{v}_s(\mathbf{p}) v_r(\mathbf{p}) = -2m \delta_{rs} \quad (32)$$

where the equation with  $v_s$  comes from replacing  $m$  with  $-m$ .

One other relation which is useful can be obtained by multiplying 9 on the left by  $\bar{u}_r \gamma^\mu$  and 17 (with subscript  $r$ ) on the right by  $\gamma^\mu v_s$  and add:

$$\bar{u}_r \gamma^\mu (\gamma^\alpha p_\alpha + m) v_s + \bar{u}_r (\gamma^\alpha p_\alpha - m) \gamma^\mu v_s = \quad (33)$$

$$\bar{u}_r (\gamma^\mu \gamma^\alpha + \gamma^\alpha \gamma^\mu) p_\alpha v_s = \quad (34)$$

$$2\bar{u}_r g^{\mu\alpha} p_\alpha v_s = 2p^\mu \bar{u}_r v_s = 0 \quad (35)$$

Since this is true for all  $p^\mu$ , we get the orthogonality condition

$$\bar{u}_r(\mathbf{p}) v_s(\mathbf{p}) = 0 \quad (36)$$

## PINGBACKS

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