DIRAC SPINORS: NORMALIZATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.8.

The Dirac equation can be written as

$$\left(i\partial\!\!\!/ - m\right)\psi(x) = 0 \tag{1}$$

This is the equation for a free particle, without any potential or interaction terms. In the rest frame of the particle, $\mathbf{p}=0$ so this equation then contains only the term involving p^0. That is

$$\left(i\gamma^0\partial_0 - m\right)\psi(x) = 0 \tag{2}$$

With a plane wave solution in the rest frame, we have

$$\psi(x) = e^{-ip\cdot x} = e^{-iEt} \tag{3}$$

so we have

$$\left(\gamma^0 E - m\right)\psi(x) = 0 \tag{4}$$

Because of the condition

$$\left(\gamma^0\right)^2 = 1 \tag{5}$$

the eigenvalues of γ^0 are ± 1, so the energy E in 4 must be

$$E = \pm m \tag{6}$$

That is, the energy can be positive or negative, so the Dirac equation suffers from the same problem as the Klein-Gordon equation in that it allows negative energies.

Because the Dirac equation involves the 4×4 gamma matrices, the solution vector $\psi(x)$ is a 4-component vector, which we can represent as

$$\psi(x) \sim \begin{cases} u_s(\mathbf{p})\, e^{-ip\cdot x} \\ v_s(\mathbf{p})\, e^{ip\cdot x} \end{cases} \tag{7}$$

where u_s and v_s are 4-component column vectors (spinors) and s labels the two independent solutions of each column vector.

Putting 7 into 1 gives us equations for the spinors:

$$\left(\not{p}-m\right)u_s\left(\mathbf{p}\right)=0 \tag{8}$$
$$\left(\not{p}+m\right)v_s\left(\mathbf{p}\right)=0 \tag{9}$$

Taking the Dirac conjugate, with

$$\overline{\psi}\equiv\psi^{\dagger}\gamma^{0} \tag{10}$$

we have from 8

$$\left(\gamma^{\mu}p_{\mu}-m\right)u_s\left(\mathbf{p}\right)=0 \tag{11}$$

Taking the hermitian conjugate and using

$$\gamma^{\mu\dagger}=\gamma^{0}\gamma^{\mu}\gamma^{0} \tag{12}$$

gives

$$u_s^{\dagger}\left(\mathbf{p}\right)\left(\gamma^{\mu\dagger}p_{\mu}-m\right)=u_s^{\dagger}\left(\mathbf{p}\right)\left(\gamma^{0}\gamma^{\mu}\gamma^{0}p_{\mu}-\gamma^{0}m\gamma^{0}\right) \tag{13}$$
$$=\overline{u}_s\left(\mathbf{p}\right)\left(\gamma^{\mu}\gamma^{0}p_{\mu}-m\gamma^{0}\right) \tag{14}$$
$$=\overline{u}_s\left(\mathbf{p}\right)\left(\not{p}-m\right)\gamma^{0} \tag{15}$$
$$=0 \tag{16}$$

Multiplying on the right by γ^{0} gives

$$\overline{u}_s\left(\mathbf{p}\right)\left(\not{p}-m\right)=0 \tag{17}$$
$$\overline{v}_s\left(\mathbf{p}\right)\left(\not{p}+m\right)=0 \tag{18}$$

where the second equation follows from a similar calculation on 9.

At this point L&P state the normalization of the spinors to be

$$u_r^{\dagger}\left(\mathbf{p}\right)u_s\left(\mathbf{p}\right)=v_r^{\dagger}\left(\mathbf{p}\right)v_s\left(\mathbf{p}\right)=2E_p\delta_{rs} \tag{19}$$
$$v_r^{\dagger}\left(\mathbf{p}\right)u_s\left(-\mathbf{p}\right)=u_r^{\dagger}\left(\mathbf{p}\right)v_s\left(-\mathbf{p}\right)=0 \tag{20}$$

We can derive a couple of useful formulas. From 8 (with subscript r) we multiply from the left by $\overline{u}_s\gamma^{\mu}$ and 17 from the right by $\gamma^{\mu}u_r$ (all u_s are assumed to be functions of $\mathbf{p}$) and add:

$$\bar{u}_s\gamma^\mu\left(\gamma^\alpha p_\alpha - m\right)u_r + \bar{u}_s\left(\gamma^\alpha p_\alpha - m\right)\gamma^\mu u_r = \tag{21}$$

$$\bar{u}_s\left(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu\right)p_\alpha u_r - 2m\bar{u}_s\gamma^\mu u_r = \tag{22}$$

$$2\bar{u}_s g^{\mu\alpha}p_\alpha u_r - 2m\bar{u}_s\gamma^\mu u_r = 0 \tag{23}$$

We therefore have

$$\bar{u}_s(\mathbf{p})\,p^\mu u_r(\mathbf{p}) = m\bar{u}_s(\mathbf{p})\,\gamma^\mu u_r(\mathbf{p}) \tag{24}$$

A similar relation for v_s follows from 9 and 18, with m replaced by $-m$:

$$\bar{v}_s(\mathbf{p})\,p^\mu v_r(\mathbf{p}) = -m\bar{v}_s(\mathbf{p})\,\gamma^\mu v_r(\mathbf{p}) \tag{25}$$

With $\mu = 0$ we have from 24 and 19

$$m\bar{u}_s(\mathbf{p})\,\gamma^0 u_r(\mathbf{p}) = mu_s^\dagger(\mathbf{p})\left(\gamma^0\right)^2 u_r(\mathbf{p}) \tag{26}$$

$$= mu_s^\dagger(\mathbf{p})\,u_r(\mathbf{p}) \tag{27}$$

$$= 2mE_p\delta_{rs} \tag{28}$$

From the LHS of 24 this is equal to

$$2mE_p\delta_{rs} = \bar{u}_s(\mathbf{p})\,p^0 u_r(\mathbf{p}) \tag{29}$$

$$= E_p\bar{u}_s(\mathbf{p})\,u_r(\mathbf{p}) \tag{30}$$

So we get an alternative form of the normalization conditions

$$\bar{u}_s(\mathbf{p})\,u_r(\mathbf{p}) = 2m\delta_{rs} \tag{31}$$

$$\bar{v}_s(\mathbf{p})\,v_r(\mathbf{p}) = -2m\delta_{rs} \tag{32}$$

where the equation with v_s comes from replacing m with $-m$.

One other relation which is useful can be obtained by multiplying 9 on the left by $\bar{u}_r\gamma^\mu$ and 17 (with subscript r) on the right by $\gamma^\mu v_s$ and add:

$$\bar{u}_r\gamma^\mu\left(\gamma^\alpha p_\alpha + m\right)v_s + \bar{u}_r\left(\gamma^\alpha p_\alpha - m\right)\gamma^\mu v_s = \tag{33}$$

$$\bar{u}_r\left(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu\right)p_\alpha v_s = \tag{34}$$

$$2\bar{u}_r g^{\mu\alpha}p_\alpha v_s = 2p^\mu\bar{u}_r v_s = 0 \tag{35}$$

Since this is true for all p^μ, we get the orthogonality condition

$$\bar{u}_r(\mathbf{p})\,v_s(\mathbf{p}) = 0 \tag{36}$$

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