DIRAC SPINORS: NORMALIZATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.8.

The Dirac equation can be written as

$$\left(i\partial\!\!\!/-m\right)\psi\left(x\right)=0 \tag{1}$$

This is the equation for a free particle, without any potential or interaction terms. In the rest frame of the particle, $\mathbf{p}=0$ so this equation then contains only the term involving p^0. That is

$$\left(i\gamma^0\partial_0-m\right)\psi\left(x\right)=0 \tag{2}$$

With a plane wave solution in the rest frame, we have

$$\psi\left(x\right)=e^{-ip\cdot x}=e^{-iEt} \tag{3}$$

so we have

$$\left(\gamma^0E-m\right)\psi\left(x\right)=0 \tag{4}$$

Because of the condition

$$\left(\gamma^0\right)^2=1 \tag{5}$$

the eigenvalues of γ^0 are ±1, so the energy E in 4 must be

$$E=\pm m \tag{6}$$

That is, the energy can be positive or negative, so the Dirac equation suffers from the same problem as the Klein-Gordon equation in that it allows negative energies.

Because the Dirac equation involves the 4×4 gamma matrices, the solution vector $\psi\left(x\right)$ is a 4-component vector, which we can represent as

$$\psi\left(x\right)\sim\begin{cases}u_s\left(\mathbf{p}\right)e^{-ip\cdot x}\\ v_s\left(\mathbf{p}\right)e^{ip\cdot x}\end{cases} \tag{7}$$

where u_s and v_s are 4-component column vectors (spinors) and s labels the two independent solutions of each column vector.

Putting 7 into 1 gives us equations for the spinors:

$$\left(\not{p}-m\right)u_s(\mathbf{p}) = 0 \tag{8}$$

$$\left(\not{p}+m\right)v_s(\mathbf{p}) = 0 \tag{9}$$

Taking the Dirac conjugate, with

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 \tag{10}$$

we have from 8

$$\left(\gamma^\mu p_\mu - m\right)u_s(\mathbf{p}) = 0 \tag{11}$$

Taking the hermitian conjugate and using

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{12}$$

gives

$$u_s^\dagger(\mathbf{p})\left(\gamma^{\mu\dagger}p_\mu - m\right) = u_s^\dagger(\mathbf{p})\left(\gamma^0\gamma^\mu\gamma^0 p_\mu - \gamma^0 m\gamma^0\right) \tag{13}$$

$$= \overline{u}_s(\mathbf{p})\left(\gamma^\mu\gamma^0 p_\mu - m\gamma^0\right) \tag{14}$$

$$= \overline{u}_s(\mathbf{p})\left(\not{p}-m\right)\gamma^0 \tag{15}$$

$$= 0 \tag{16}$$

Multiplying on the right by γ^0 gives

$$\overline{u}_s(\mathbf{p})\left(\not{p}-m\right) = 0 \tag{17}$$

$$\overline{v}_s(\mathbf{p})\left(\not{p}+m\right) = 0 \tag{18}$$

where the second equation follows from a similar calculation on 9.

At this point L&P state the normalization of the spinors to be

$$u_r^\dagger(\mathbf{p})\,u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p})\,v_s(\mathbf{p}) = 2E_p\delta_{rs} \tag{19}$$

$$v_r^\dagger(\mathbf{p})\,u_s(-\mathbf{p}) = u_r^\dagger(\mathbf{p})\,v_s(-\mathbf{p}) = 0 \tag{20}$$

We can derive a couple of useful formulas. From 8 (with subscript r) we multiply from the left by $\overline{u}_s\gamma^\mu$ and 17 from the right by $\gamma^\mu u_r$ (all u_s are assumed to be functions of $\mathbf{p}$) and add:

$$\overline{u}_s\gamma^\mu\left(\gamma^\alpha p_\alpha - m\right)u_r + \overline{u}_s\left(\gamma^\alpha p_\alpha - m\right)\gamma^\mu u_r = \tag{21}$$
$$\overline{u}_s\left(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu\right)p_\alpha u_r - 2m\overline{u}_s\gamma^\mu u_r = \tag{22}$$
$$2\overline{u}_s g^{\mu\alpha}p_\alpha u_r - 2m\overline{u}_s\gamma^\mu u_r = 0 \tag{23}$$

We therefore have

$$\overline{u}_s(\mathbf{p})\,p^\mu u_r(\mathbf{p}) = m\overline{u}_s(\mathbf{p})\,\gamma^\mu u_r(\mathbf{p}) \tag{24}$$

A similar relation for v_s follows from 9 and 18, with m replaced by $-m$:

$$\overline{v}_s(\mathbf{p})\,p^\mu v_r(\mathbf{p}) = -m\overline{v}_s(\mathbf{p})\,\gamma^\mu v_r(\mathbf{p}) \tag{25}$$

With $\mu = 0$ we have from 24 and 19

$$m\overline{u}_s(\mathbf{p})\,\gamma^0 u_r(\mathbf{p}) = mu_s^\dagger(\mathbf{p})\left(\gamma^0\right)^2 u_r(\mathbf{p}) \tag{26}$$
$$= mu_s^\dagger(\mathbf{p})\,u_r(\mathbf{p}) \tag{27}$$
$$= 2mE_p\delta_{rs} \tag{28}$$

From the LHS of 24 this is equal to

$$2mE_p\delta_{rs} = \overline{u}_s(\mathbf{p})\,p^0 u_r(\mathbf{p}) \tag{29}$$
$$= E_p\overline{u}_s(\mathbf{p})\,u_r(\mathbf{p}) \tag{30}$$

So we get an alternative form of the normalization conditions

$$\overline{u}_s(\mathbf{p})\,u_r(\mathbf{p}) = 2m\delta_{rs} \tag{31}$$
$$\overline{v}_s(\mathbf{p})\,v_r(\mathbf{p}) = -2m\delta_{rs} \tag{32}$$

where the equation with v_s comes from replacing m with $-m$.

One other relation which is useful can be obtained by multiplying 9 on the left by $\overline{u}_r\gamma^\mu$ and 17 (with subscript r) on the right by $\gamma^\mu v_s$ and add:

$$\overline{u}_r\gamma^\mu\left(\gamma^\alpha p_\alpha + m\right)v_s + \overline{u}_r\left(\gamma^\alpha p_\alpha - m\right)\gamma^\mu v_s = \tag{33}$$
$$\overline{u}_r\left(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu\right)p_\alpha v_s = \tag{34}$$
$$2\overline{u}_r g^{\mu\alpha}p_\alpha v_s = 2p^\mu\overline{u}_r v_s = 0 \tag{35}$$

Since this is true for all p^μ, we get the orthogonality condition

$$\overline{u}_r(\mathbf{p})\,v_s(\mathbf{p}) = 0 \tag{36}$$

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