DIRAC SPINORS: GORDON IDENTITY

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Post date: 11 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.9.

The Dirac spinors satisfy the relations

$$\left(\not{p}-m\right)u_s(\mathbf{p}) = 0 \tag{1}$$

$$\left(\not{p}+m\right)v_s(\mathbf{p}) = 0 \tag{2}$$

$$\overline{u}_s(\mathbf{p})\left(\not{p}-m\right) = 0 \tag{3}$$

$$\overline{v}_s(\mathbf{p})\left(\not{p}+m\right) = 0 \tag{4}$$

where a barred spinor is defined by

$$\overline{\psi} \equiv \psi^{\dagger}\gamma^{0} \tag{5}$$

We've seen that these spinors satisfy the identities:

$$\overline{u}_s(\mathbf{p})\,p^{\mu}u_r(\mathbf{p}) = m\overline{u}_s(\mathbf{p})\,\gamma^{\mu}u_r(\mathbf{p}) \tag{6}$$

$$\overline{v}_s(\mathbf{p})\,p^{\mu}v_r(\mathbf{p}) = -m\overline{v}_s(\mathbf{p})\,\gamma^{\mu}v_r(\mathbf{p}) \tag{7}$$

The Gordon identity relates spinors at different momenta. We can derive it using the properties of the gamma matrices. To streamline the notation, I'll use the following definitions:

$$u \equiv u_s(\mathbf{p}) \tag{8}$$

$$u' \equiv u_s\left(\mathbf{p}'\right) \tag{9}$$

with a similar notation for v and v'. The subscript s can be 1 or 2.

Multiply 1 on the left by $\overline{u}'\gamma^{\mu}$:

$$\overline{u}'\gamma^{\mu}\gamma^{\alpha}p_{\alpha}u = m\overline{u}'\gamma^{\mu}u \tag{10}$$

Now multiply 1 (with momentum $\mathbf{p}'$) on the right by $\gamma^{\mu}u$:

$$\overline{u}'\gamma^{\alpha}p'_{\alpha}\gamma^{\mu}u = m\overline{u}'\gamma^{\mu}u \tag{11}$$

Now add these two equations:

$$\overline{u}'\left(\gamma^{\mu}\gamma^{\alpha}p_{\alpha}+\gamma^{\alpha}\gamma^{\mu}p'_{\alpha}\right)u=2m\overline{u}'\gamma^{\mu}u \tag{12}$$

Using the anticommutator

$$\{\gamma^{\mu},\gamma^{\alpha}\}=2g^{\mu\alpha} \tag{13}$$

Now consider (using the definition of $\sigma^{\mu\alpha}$):

$$-i\sigma^{\mu\alpha}q_{\alpha}=-i\frac{i}{2}\left(\gamma^{\mu}\gamma^{\alpha}-\gamma^{\alpha}\gamma^{\mu}\right)\left(p_{\alpha}-p'_{\alpha}\right) \tag{14}$$

$$=\frac{1}{2}\left(\gamma^{\mu}\gamma^{\alpha}-2g^{\alpha\mu}+\gamma^{\mu}\gamma^{\alpha}\right)p_{\alpha}+ \tag{15}$$

$$\frac{1}{2}\left(\gamma^{\alpha}\gamma^{\mu}-2g^{\mu\alpha}+\gamma^{\alpha}\gamma^{\mu}\right)p'_{\alpha} \tag{16}$$

$$=\gamma^{\mu}\gamma^{\alpha}p_{\alpha}+\gamma^{\alpha}\gamma^{\mu}p'_{\alpha}-\left(p+p'\right)^{\mu} \tag{17}$$

The first two terms are the same as the quantity in parentheses in 12 so, substituting this back into 12 we have

$$\overline{u}'\gamma^{\mu}u=\frac{1}{2m}\overline{u}'\left[\left(p+p'\right)^{\mu}-i\sigma^{\mu\alpha}q_{\alpha}\right]u \tag{18}$$

We can get a similar expression involving v by replacing m by $-m$:

$$\overline{v}'\gamma^{\mu}v=-\frac{1}{2m}\overline{v}'\left[\left(p+p'\right)^{\mu}-i\sigma^{\mu\alpha}q_{\alpha}\right]v \tag{19}$$

This is the Gordon identity.