DIRAC SPINORS: PRODUCTS OF SPINORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.10.

The Dirac spinors satisfy the relations

$$\left(\not p-m\right)u_{s}\left(\mathbf{p}\right)=0 \tag{1}$$

$$\left(\not p+m\right)v_{s}\left(\mathbf{p}\right)=0 \tag{2}$$

$$\overline{u}_{s}\left(\mathbf{p}\right)\left(\not p-m\right)=0 \tag{3}$$

$$\overline{v}_{s}\left(\mathbf{p}\right)\left(\not p+m\right)=0 \tag{4}$$

where a barred spinor is defined by

$$\overline{\psi}\equiv\psi^{\dagger}\gamma^{0} \tag{5}$$

Each spinor is a 4-component column vector, while the barred versions are each a 4-component row vector. Therefore, if we form the product $u_{s}\overline{u}_{s}$ we get a 4×4 matrix. If we sum over the two s indexes, we can prove the identities

$$\sum_{s}u_{s}\left(\mathbf{p}\right)\overline{u}_{s}\left(\mathbf{p}\right)=\not p+m \tag{6}$$

$$\sum_{s}v_{s}\left(\mathbf{p}\right)\overline{v}_{s}\left(\mathbf{p}\right)=\not p-m \tag{7}$$

We can show this by applying both sides of each equation to each of the four basis vectors u_{s} and v_{s}, where s can be $+$ or $-$ in each case. For the LHS, we can use the results

$$\overline{u}_{s}\left(\mathbf{p}\right)u_{r}\left(\mathbf{p}\right)=2m\delta_{rs} \tag{8}$$

$$\overline{v}_{s}\left(\mathbf{p}\right)v_{r}\left(\mathbf{p}\right)=-2m\delta_{rs} \tag{9}$$

For example (all spinors are functions of $\mathbf{p}$):

$$\left[\sum_s u_s\bar{u}_s\right]u_- = u_+\bar{u}_+u_- + u_-\bar{u}_-u_- \tag{10}$$
$$= 0 + 2mu_- \tag{11}$$
$$= 2mu_- \tag{12}$$

The RHS is, using 1

$$\left(\not{p}+m\right)u_- = \left(\not{p}-m+2m\right)u_- \tag{13}$$
$$= \left(\not{p}-m\right)u_- + 2mu_- \tag{14}$$
$$= 2mu_- \tag{15}$$

Thus $\sum_s \bar{u}_s(\mathbf{p})\,u_s(\mathbf{p})$ has the same effect on u_- as $\not{p}+m$. The same argument works for u_+.

For v_-, we can use the relation

$$\bar{u}_r(\mathbf{p})\,v_s(\mathbf{p}) = 0 \tag{16}$$

Thus

$$\sum_s u_s\bar{u}_s v_s = 0 \tag{17}$$

and from 2

$$\left(\not{p}+m\right)v_s(\mathbf{p}) = 0 \tag{18}$$

Thus 6 is true when both sides are applied to all four basis vectors.

The argument for 7 is essentially the same. For example, for the LHS:

$$\left[\sum_s v_s\bar{v}_s\right]v_- = v_+\bar{v}_+v_- + v_-\bar{v}_-v_- \tag{19}$$
$$= 0 - 2mv_- \tag{20}$$
$$= -2mv_- \tag{21}$$

and for the RHS

$$\left(\not{p}-m\right)v_- = \left(\not{p}+m-2m\right)v_- \tag{22}$$
$$= -2mv_- \tag{23}$$

Since both sides of both equations 6 and 7 have the same effect on all four basis vectors, the operators are equivalent.

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