# EXPLICIT SOLUTIONS OF DIRAC EQUATION IN DIRAC-PAULI REPRESENTATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.11.

The solution of the Dirac equation for a free particle is

$$\psi(x) \sim \begin{cases} u_s(\mathbf{p})\, e^{-ip\cdot x} \\ v_s(\mathbf{p})\, e^{ip\cdot x} \end{cases} \tag{1}$$

where $u_s$ and $v_s$ are 4-component spinors and $s = +$ or $-$. These spinors satisfy

$$\left(\not{p} - m\right) u_s(\mathbf{p}) = 0 \tag{2}$$

$$\left(\not{p} + m\right) v_s(\mathbf{p}) = 0 \tag{3}$$

To find explicit forms for the spinors, we need an explicit representation of the gamma matrices. One such representation is the Dirac-Pauli representation in which the matrices are given by

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{4}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{5}$$

where the $\sigma^i$ are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

Each of the entries in 4 and 5 is a $2\times 2$ matrix, while the entries in the Pauli matrices are ordinary numbers. The four-vector $p$ has the general form

$$p = (E_p, \mathbf{p}) \tag{7}$$

where

$$E_p = \sqrt{\mathbf{p}^2 + m^2} \tag{8}$$

With the particular representation of the gamma matrices, we have

$$\not{p} - m = \gamma_0 E_p - \gamma \cdot \mathbf{p} - mI \tag{9}$$

$$= \begin{bmatrix} E_p - m & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -E_p - m \end{bmatrix} \tag{10}$$

Plugging this into 2 results in

$E_p - m$ is actually $(E_p - m) I$, with $I$ being the $2 \times 2$ identity matrix.

$$(E_p - m)\,\phi_t - \sigma \cdot \mathbf{p}\phi_b = 0 \tag{11}$$

$$\sigma \cdot \mathbf{p}\phi_t - (E_p + m)\,\phi_b = 0 \tag{12}$$

where $\phi_t$ is a column vector with the top two components of $u_s$ and $\phi_b$ contains the bottom two components of $u_s$. Each term in the equations 11 is a two-component vector, since it is the sum of terms containing the product of a $2 \times 2$ matrix and a 2-component column vector.

To find $u_s$, we assume that everything in 11 is specified except for $\phi_t$ and $\phi_b$, so these two equations comprise a system of two equations in two unknowns. This system has a solution if the determinant of the coefficients is zero, which we can verify.

$$\begin{vmatrix} E_p - m & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -E_p - m \end{vmatrix} = -(E_p - m)(E_p + m) + (\sigma \cdot \mathbf{p})^2 \tag{13}$$

$$= -E_p^2 + m^2 + (\sigma \cdot \mathbf{p})^2 \tag{14}$$

$$= -\mathbf{p}^2 I + (\sigma \cdot \mathbf{p})^2 \tag{15}$$

The last term can be worked out using 6:

$$(\sigma \cdot \mathbf{p})^2 = \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix}^2 \tag{16}$$

$$= \begin{bmatrix} \mathbf{p}^2 & 0 \\ 0 & \mathbf{p}^2 \end{bmatrix} \tag{17}$$

$$= \mathbf{p}^2 I \tag{18}$$

Plugging this into 15 we have

$$\begin{vmatrix} E_p - m & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -E_p - m \end{vmatrix} = 0 \tag{19}$$

as required.

From the second of 11 we have

$$\phi_b = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \phi_t \tag{20}$$

so the solution is

$$u_{\pm}(\mathbf{p}) = A \begin{bmatrix} \chi_{\pm} \\ \frac{\sigma \cdot \mathbf{p}}{E_p + m} \chi_{\pm} \end{bmatrix} \tag{21}$$

where $A$ is a constant determined by the normalization condition

$$u_r^{\dagger}(\mathbf{p}) u_s(\mathbf{p}) = v_r^{\dagger}(\mathbf{p}) v_s(\mathbf{p}) = 2E_p \delta_{rs} \tag{22}$$

The 2-component vectors $\chi_{\pm}$ can in fact be any pair of linearly independent, normalized vectors. In this case, the normalization constant $A$ is

$$A = \sqrt{E_p + m} \tag{23}$$

For the particular solution in L&P they choose

$$\chi_{+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{24}$$

$$\chi_{-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{25}$$

which gives

$$u_{\pm}(\mathbf{p}) = \sqrt{E_p + m} \begin{bmatrix} \chi_{\pm} \\ \frac{\sigma \cdot \mathbf{p}}{E_p + m} \chi_{\pm} \end{bmatrix} \tag{26}$$

We can go through a similar argument to find $v_{\pm}$, and with the same choice of $\chi_{\pm}$, we have

$$v_{\pm}(\mathbf{p}) = \pm \sqrt{E_p + m} \begin{bmatrix} \frac{\sigma \cdot \mathbf{p}}{E_p + m} \chi_{\mp} \\ \chi_{\mp} \end{bmatrix} \tag{27}$$

The swapping of $\chi_{+}$ with $\chi_{-}$ is just a convention.

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