

EXPLICIT SOLUTIONS OF DIRAC EQUATION IN DIRAC-PAULI REPRESENTATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.11.

The solution of the Dirac equation for a free particle is

$$\psi(x) \sim \begin{cases} u_s(\mathbf{p}) e^{-ip \cdot x} \\ v_s(\mathbf{p}) e^{ip \cdot x} \end{cases} \quad (1)$$

where u_s and v_s are 4-component spinors and $s = +$ or $-$. These spinors satisfy

$$(\not{p} - m) u_s(\mathbf{p}) = 0 \quad (2)$$

$$(\not{p} + m) v_s(\mathbf{p}) = 0 \quad (3)$$

To find explicit forms for the spinors, we need an explicit representation of the gamma matrices. One such representation is the Dirac-Pauli representation in which the matrices are given by

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (4)$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad (5)$$

where the σ^i are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (6)$$

Each of the entries in 4 and 5 is a 2×2 matrix, while the entries in the Pauli matrices are ordinary numbers. The four-vector p has the general form

$$p = (E_p, \mathbf{p}) \quad (7)$$

where

$$E_p = \sqrt{\mathbf{p}^2 + m^2} \quad (8)$$

With the particular representation of the gamma matrices, we have

$$\not{p} - m = \gamma_0 E_p - \boldsymbol{\gamma} \cdot \mathbf{p} - mI \quad (9)$$

$$= \begin{bmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p - m \end{bmatrix} \quad (10)$$

Plugging this into 2 results in

$$(E_p - m)\phi_t - \boldsymbol{\sigma} \cdot \mathbf{p}\phi_b = 0 \quad (11)$$

$$\boldsymbol{\sigma} \cdot \mathbf{p}\phi_t - (E_p + m)\phi_b = 0 \quad (12)$$

$E_p - m$ is actually $(E_p - m)I$, with I being the 2×2 identity matrix.

where ϕ_t is a column vector with the top two components of u_s and ϕ_b contains the bottom two components of u_s . Each term in the equations 11 is a two-component vector, since it is the sum of terms containing the product of a 2×2 matrix and a 2-component column vector.

To find u_s , we assume that everything in 11 is specified except for ϕ_t and ϕ_b , so these two equations comprise a system of two equations in two unknowns. This system has a solution if the determinant of the coefficients is zero, which we can verify.

$$\begin{vmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p - m \end{vmatrix} = -(E_p - m)(E_p + m) + (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \quad (13)$$

$$= -E_p^2 + m^2 + (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \quad (14)$$

$$= -\mathbf{p}^2 I + (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \quad (15)$$

The last term can be worked out using 6:

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix}^2 \quad (16)$$

$$= \begin{bmatrix} \mathbf{p}^2 & 0 \\ 0 & \mathbf{p}^2 \end{bmatrix} \quad (17)$$

$$= \mathbf{p}^2 I \quad (18)$$

Plugging this into 15 we have

$$\begin{vmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p - m \end{vmatrix} = 0 \quad (19)$$

as required.

From the second of 11 we have

$$\phi_b = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \phi_t \quad (20)$$

so the solution is

$$u_{\pm}(\mathbf{p}) = A \left[\begin{array}{c} \chi_{\pm} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \chi_{\pm} \end{array} \right] \quad (21)$$

where A is a constant determined by the normalization condition

$$u_r^{\dagger}(\mathbf{p}) u_s(\mathbf{p}) = v_r^{\dagger}(\mathbf{p}) v_s(\mathbf{p}) = 2E_p \delta_{rs} \quad (22)$$

The 2-component vectors χ_{\pm} can in fact be any pair of linearly independent, normalized vectors. In this case, the normalization constant A is

$$A = \sqrt{E_p + m} \quad (23)$$

For the particular solution in L&P they choose

$$\chi_+ = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \quad (24)$$

$$\chi_- = \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \quad (25)$$

which gives

$$u_{\pm}(\mathbf{p}) = \sqrt{E_p + m} \left[\begin{array}{c} \chi_{\pm} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \chi_{\pm} \end{array} \right] \quad (26)$$

We can go through a similar argument to find v_{\pm} , and with the same choice of χ_{\pm} , we have

$$v_{\pm}(\mathbf{p}) = \pm \sqrt{E_p + m} \left[\begin{array}{c} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \chi_{\mp} \\ \chi_{\mp} \end{array} \right] \quad (27)$$

The swapping of χ_+ with χ_- is just a convention.

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