EXPLICIT SOLUTIONS OF DIRAC EQUATION FROM PARTICLE'S REST FRAME

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.12.

The solution of the Dirac equation for a free particle is

$$\psi(x) \sim \begin{cases} u_s(\mathbf{p})\, e^{-ip\cdot x} \\ v_s(\mathbf{p})\, e^{ip\cdot x} \end{cases} \tag{1}$$

where u_s and v_s are 4-component spinors and $s = +$ or $-$. These spinors satisfy

$$\left(\not{p} - m\right) u_s(\mathbf{p}) = 0 \tag{2}$$

$$\left(\not{p} + m\right) v_s(\mathbf{p}) = 0 \tag{3}$$

To find explicit forms for the spinors, we need an explicit representation of the gamma matrices. One such representation is the Dirac-Pauli representation in which the matrices are given by

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{4}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{5}$$

where the σ^i are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

Each of the entries in 4 and 5 is a 2×2 matrix, while the entries in the Pauli matrices are ordinary numbers. The four-vector p has the general form

$$p = (E_p, \mathbf{p}) \tag{7}$$

where

$$E_p = \sqrt{\mathbf{p}^2 + m^2} \tag{8}$$

Earlier, we saw how to obtain explicit solutions for the Dirac spinors using these equations. Here we look at an alternative derivation. In the particle's rest frame $\mathbf{p} = 0$ and $E_p = m$, and we can choose the following spinors as solutions:

$$u_+(\mathbf{0}) = \sqrt{2m}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

$$u_-(\mathbf{0}) = \sqrt{2m}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$v_+(\mathbf{0}) = \sqrt{2m}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{11}$$

$$v_-(\mathbf{0}) = \sqrt{2m}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \tag{12}$$

These spinors satisfy the normalization condition for a particle at rest:

$$u_r^\dagger(\mathbf{p})\, u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p})\, v_s(\mathbf{p}) = 2E_p\delta_{rs} = 2m\delta_{rs} \tag{13}$$

We can use these solutions to generate the solutions for arbitrary 3-momentum $\mathbf{p}$ by showing that $\left(\not{p} + m\right) u_\pm(\mathbf{0})$ is a solution of 2 and $\left(\not{p} - m\right) v_\pm(\mathbf{0})$ is a solution to 3. We can do this by direct substitution, but first we need the result for two four-vectors a and b:

$$\begin{aligned} \not{a}\not{b} &= \gamma^\mu a_\mu \gamma^\nu b_\nu && (14)\\ &= \gamma^\mu \gamma^\nu a_\mu b_\nu && (15)\\ &= \left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right) a_\mu b_\nu && (16)\\ &= 2a^\mu b_\mu - \not{b}\not{a} && (17) \end{aligned}$$

In the particular case where $a = b$ we have

$$\not{a}\not{a} = a^\mu a_\mu = a^2 \tag{18}$$

We now have from 2

$$\left(\not{p}-m\right)\left(\not{p}+m\right)u_{\pm}\left(\mathbf{0}\right) = \left(\not{p}\not{p}-m^{2}\right)u_{\pm}\left(\mathbf{0}\right) \tag{19}$$

$$= \left(p^{2}-m^{2}\right)u_{\pm}\left(\mathbf{0}\right) \tag{20}$$

$$= 0 \tag{21}$$

where the last line follows because, for $\mathbf{p}=0$, $p^2=m^2$.

To get the actual solution, we need to apply $\not{p}+m$ to $u_{\pm}\left(\mathbf{0}\right)$ in the Dirac-Pauli representation. We've seen that, in this representation

$$\not{p}-m = \gamma_0 E_p - \gamma\cdot\mathbf{p} - mI \tag{22}$$

$$= \begin{bmatrix} E_p - m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E_p - m \end{bmatrix} \tag{23}$$

so we also have

Each entry in these matrices is itself a 2×2 matrix.

$$\not{p}+m = \gamma_0 E_p - \gamma\cdot\mathbf{p} + mI \tag{24}$$

$$= \begin{bmatrix} E_p + m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E_p + m \end{bmatrix} \tag{25}$$

Therefore we have

$$\left(\not{p}+m\right)u_{+}\left(\mathbf{0}\right) = B\begin{bmatrix} E_p+m \\ 0 \\ \left(\sigma\cdot\mathbf{p}\right)_{11} \\ \left(\sigma\cdot\mathbf{p}\right)_{21} \end{bmatrix} \tag{26}$$

$$\left(\not{p}+m\right)u_{-}\left(\mathbf{0}\right) = B\begin{bmatrix} 0 \\ E_p+m \\ \left(\sigma\cdot\mathbf{p}\right)_{12} \\ \left(\sigma\cdot\mathbf{p}\right)_{22} \end{bmatrix} \tag{27}$$

where BS is a normalization constant. To compare these with the solutions found earlier, we note that

$$\sigma\cdot\mathbf{p} = \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix} \tag{28}$$

and the earlier solutions were

$$u_{\pm}\left(\mathbf{p}\right) = \sqrt{E_p+m}\begin{bmatrix} \chi_{\pm} \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_{\pm} \end{bmatrix} \tag{29}$$

with

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{30}$$

$$\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{31}$$

Thus our present solutions 26 match the earlier ones if the momentum is $\mathbf{p} = \left(0,0,p^3\right)$, that is, it's entirely along the z direction, and we normalize the solutions correctly.

We can make the solutions match if we write

$$u_+(\mathbf{0}) = \sqrt{2m}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{2m}\begin{bmatrix} \chi_+ \\ 0 \end{bmatrix} \tag{32}$$

where each entry on the RHS is now a 2-component column vector. If we now apply 25 to this and multiply the elements as though they are single matrix elements, we get

$$\left(\not{p}+m\right)u_+(\mathbf{0}) = B\begin{bmatrix} \left(E_p+m\right)\chi_+ \\ \sigma\cdot\mathbf{p}\chi_+ \end{bmatrix} \tag{33}$$

which matches 29, after normalization. I'm not sure this is allowable, however.

We can do a similar calculation to get $v_\pm$:

$$\left(\not{p}+m\right)\left(\not{p}-m\right)v_\pm(\mathbf{0}) = \left(\not{p}\not{p}-m^2\right)v_\pm(\mathbf{0}) \tag{34}$$

$$= \left(p^2-m^2\right)v_\pm(\mathbf{0}) \tag{35}$$

$$= 0 \tag{36}$$

Therefore

$$\left(\not{p}-m\right)v_+(\mathbf{0}) = B\begin{bmatrix} E_p-m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E_p-m \end{bmatrix}\begin{bmatrix} 0 \\ \chi_- \end{bmatrix} \tag{37}$$

$$= -B\begin{bmatrix} \sigma\cdot\mathbf{p}\chi_- \\ \left(E_p+m\right)\chi_- \end{bmatrix} \tag{38}$$

$$\left(\not{p}-m\right)v_-(\mathbf{0}) = B\begin{bmatrix} E_p-m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E_p-m \end{bmatrix}\begin{bmatrix} 0 \\ -\chi_+ \end{bmatrix} \tag{39}$$

$$= B\begin{bmatrix} \sigma\cdot\mathbf{p}\chi_+ \\ \left(E_p+m\right)\chi_+ \end{bmatrix} \tag{40}$$

These again agree with the earlier solutions after proper normalization:

$$v_{\pm}(\mathbf{p}) = \pm\sqrt{E_p+m}\left[\begin{array}{c}\frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_{\mp}\\ \chi_{\mp}\end{array}\right] \tag{41}$$

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