

## EXPLICIT SOLUTIONS OF DIRAC EQUATION FROM PARTICLE'S REST FRAME

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Post date: 15 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.12.

The solution of the Dirac equation for a free particle is

$$\psi(x) \sim \begin{cases} u_s(\mathbf{p}) e^{-ip \cdot x} \\ v_s(\mathbf{p}) e^{ip \cdot x} \end{cases} \quad (1)$$

where  $u_s$  and  $v_s$  are 4-component spinors and  $s = +$  or  $-$ . These spinors satisfy

$$(\not{p} - m) u_s(\mathbf{p}) = 0 \quad (2)$$

$$(\not{p} + m) v_s(\mathbf{p}) = 0 \quad (3)$$

To find explicit forms for the spinors, we need an explicit representation of the gamma matrices. One such representation is the Dirac-Pauli representation in which the matrices are given by

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (4)$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad (5)$$

where the  $\sigma^i$  are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (6)$$

Each of the entries in 4 and 5 is a  $2 \times 2$  matrix, while the entries in the Pauli matrices are ordinary numbers. The four-vector  $p$  has the general form

$$p = (E_p, \mathbf{p}) \quad (7)$$

where

$$E_p = \sqrt{\mathbf{p}^2 + m^2} \quad (8)$$

Earlier, we saw how to obtain explicit solutions for the Dirac spinors using these equations. Here we look at an alternative derivation. In the particle's rest frame  $\mathbf{p} = 0$  and  $E_p = m$ , and we can choose the following spinors as solutions:

$$u_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

$$u_-(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (10)$$

$$v_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (11)$$

$$v_-(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad (12)$$

These spinors satisfy the normalization condition for a particle at rest:

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = 2E_p \delta_{rs} = 2m \delta_{rs} \quad (13)$$

We can use these solutions to generate the solutions for arbitrary 3-momentum  $\mathbf{p}$  by showing that  $(\not{p} + m) u_\pm(\mathbf{0})$  is a solution of 2 and  $(\not{p} - m) v_\pm(\mathbf{0})$  is a solution to 3. We can do this by direct substitution, but first we need the result for two four-vectors  $a$  and  $b$ :

$$\not{a} \not{b} = \gamma^\mu a_\mu \gamma^\nu b_\nu \quad (14)$$

$$= \gamma^\mu \gamma^\nu a_\mu b_\nu \quad (15)$$

$$= (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) a_\mu b_\nu \quad (16)$$

$$= 2a^\mu b_\mu - \not{b} \not{a} \quad (17)$$

In the particular case where  $a = b$  we have

$$\not{a} \not{a} = a^\mu a_\mu = a^2 \quad (18)$$

We now have from 2

$$(\not{p} - m)(\not{p} + m)u_{\pm}(\mathbf{0}) = (\not{p}\not{p} - m^2)u_{\pm}(\mathbf{0}) \quad (19)$$

$$= (p^2 - m^2)u_{\pm}(\mathbf{0}) \quad (20)$$

$$= 0 \quad (21)$$

where the last line follows because, for  $\mathbf{p} = 0$ ,  $p^2 = m^2$ .

To get the actual solution, we need to apply  $\not{p} + m$  to  $u_{\pm}(\mathbf{0})$  in the Dirac-Pauli representation. We've seen that, in this representation

$$\not{p} - m = \gamma_0 E_p - \boldsymbol{\gamma} \cdot \mathbf{p} - mI \quad (22)$$

$$= \begin{bmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p - m \end{bmatrix} \quad (23)$$

so we also have

$$\not{p} + m = \gamma_0 E_p - \boldsymbol{\gamma} \cdot \mathbf{p} + mI \quad (24)$$

$$= \begin{bmatrix} E_p + m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p + m \end{bmatrix} \quad (25)$$

Each entry in these matrices is itself a  $2 \times 2$  matrix.

Therefore we have

$$(\not{p} + m)u_+(\mathbf{0}) = B \begin{bmatrix} E_p + m \\ 0 \\ (\boldsymbol{\sigma} \cdot \mathbf{p})_{11} \\ (\boldsymbol{\sigma} \cdot \mathbf{p})_{21} \end{bmatrix} \quad (26)$$

$$(\not{p} + m)u_-(\mathbf{0}) = B \begin{bmatrix} 0 \\ E_p + m \\ (\boldsymbol{\sigma} \cdot \mathbf{p})_{12} \\ (\boldsymbol{\sigma} \cdot \mathbf{p})_{22} \end{bmatrix} \quad (27)$$

where  $BS$  is a normalization constant. To compare these with the solutions found earlier, we note that

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix} \quad (28)$$

and the earlier solutions were

$$u_{\pm}(\mathbf{p}) = \sqrt{E_p + m} \begin{bmatrix} \chi_{\pm} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \chi_{\pm} \end{bmatrix} \quad (29)$$

with

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (30)$$

$$\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (31)$$

Thus our present solutions 26 match the earlier ones if the momentum is  $\mathbf{p} = (0, 0, p^3)$ , that is, it's entirely along the  $z$  direction, and we normalize the solutions correctly.

We can make the solutions match if we write

$$u_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{2m} \begin{bmatrix} \chi_+ \\ 0 \end{bmatrix} \quad (32)$$

where each entry on the RHS is now a 2-component column vector. If we now apply 25 to this and multiply the elements as though they are single matrix elements, we get

$$(\not{p} + m) u_+(\mathbf{0}) = B \begin{bmatrix} (E_p + m) \chi_+ \\ \boldsymbol{\sigma} \cdot \mathbf{p} \chi_+ \end{bmatrix} \quad (33)$$

which matches 29, after normalization. I'm not sure this is allowable, however.

We can do a similar calculation to get  $v_{\pm}$ :

$$(\not{p} + m) (\not{p} - m) v_{\pm}(\mathbf{0}) = (\not{p} \not{p} - m^2) v_{\pm}(\mathbf{0}) \quad (34)$$

$$= (p^2 - m^2) v_{\pm}(\mathbf{0}) \quad (35)$$

$$= 0 \quad (36)$$

Therefore

$$(\not{p} - m) v_+(\mathbf{0}) = B \begin{bmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p - m \end{bmatrix} \begin{bmatrix} 0 \\ \chi_- \end{bmatrix} \quad (37)$$

$$= -B \begin{bmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} \chi_- \\ (E_p + m) \chi_- \end{bmatrix} \quad (38)$$

$$(\not{p} - m) v_-(\mathbf{0}) = B \begin{bmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -E_p - m \end{bmatrix} \begin{bmatrix} 0 \\ -\chi_+ \end{bmatrix} \quad (39)$$

$$= B \begin{bmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} \chi_+ \\ (E_p + m) \chi_+ \end{bmatrix} \quad (40)$$

These again agree with the earlier solutions after proper normalization:

$$v_{\pm}(\mathbf{p}) = \pm \sqrt{E_p + m} \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \chi_{\mp} \\ \chi_{\mp} \end{bmatrix} \quad (41)$$

## PINGBACKS

Pingback: Energy projection operators in the Dirac equation

Pingback: Spin projection operators in the Dirac equation

Pingback: Spin projection operators in the Dirac equation for a particle at rest