HELICITY OPERATOR IN THE DIRAC EQUATION (LAHIRI & PAL)

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.14.

We'll revisit here the helicity operator in the Dirac equation. Helicity is defined as the component of the spin parallel to the particle's momentum. For a massive particle (that is, one travelling at less than the speed of light), the spin can be oriented at some angle other than parallel to the momentum, so the helicity is defined as the projection of the spin onto the momentum. The operator is defined as

$$\Sigma_{\mathbf{p}} \equiv \frac{\Sigma \cdot \mathbf{p}}{\mathfrak{p}} \tag{1}$$

where Σ is a 3-vector defined in terms of the spin matrices as

$$\Sigma \equiv \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right) \tag{2}$$

which amounts to the components of spin along the x, y and z axes. The quantity $\mathfrak{p}$ is the magnitude of the 3-momentum:

$$\mathfrak{p} \equiv |\mathbf{p}| \tag{3}$$

The operator $\Sigma_{\mathbf{p}}$ has eigenvalues of ± 1 as we can see as follows. Any matrix A whose square is the unit matrix has eigenvalues of ± 1, since if v is an eigenvector of A, then

$$\begin{aligned}
Av &= \lambda v && (4)\\
A^2 v &= A\lambda v && (5)\\
&= \lambda A v && (6)\\
&= \lambda^2 v && (7)\\
&= v && (8)
\end{aligned}$$

so that

$$\lambda = \pm 1 \tag{9}$$

Thus we need to show that

$$\Sigma_{\mathbf{p}}^2 = I \tag{10}$$

We can prove this in general (without recourse to any particular representation of the γ^μ matrices) as follows. We have

$$\Sigma_{\mathbf{p}}^2 = \frac{1}{\mathfrak{p}^2}\left(\sigma^{23}p^1 + \sigma^{31}p^2 + \sigma^{12}p^3\right)^2 \tag{11}$$

From the definition of σ^{ij} and the properties of the γ^μ we have (taking $i \neq j$):

$$\begin{aligned}
\left(\sigma^{ij}\right)^2 &= \left(i\gamma^i\gamma^j\right)^2 &(12)\\
&= -\gamma^i\gamma^j\gamma^i\gamma^j &(13)\\
&= \gamma^j\gamma^i\gamma^i\gamma^j &(14)\\
&= -\gamma^j\gamma^j &(15)\\
&= +1 &(16)
\end{aligned}$$

Also:

$$\begin{aligned}
\left\{\sigma^{23},\sigma^{31}\right\} &= -\gamma^2\gamma^3\gamma^3\gamma^1 - \gamma^3\gamma^1\gamma^2\gamma^3 &(17)\\
&= \gamma^2\gamma^1 - \gamma^1\gamma^2\left(\gamma^3\right)^2 &(18)\\
&= \gamma^2\gamma^1 + \gamma^1\gamma^2 &(19)\\
&= 0 &(20)
\end{aligned}$$

The other anticommutators required follow from similar calculations, so all of them are zero. Returning to 11 we have

$$\begin{aligned}
\Sigma_{\mathbf{p}}^2 = \frac{1}{\mathfrak{p}^2}\Big[&\left(\sigma^{23}p^1\right)^2 + \left(\sigma^{31}p^2\right)^2 + \left(\sigma^{12}p^3\right)^2 + &(21)\\
&p^1p^2\left\{\sigma^{23},\sigma^{31}\right\} + p^3p^1\left\{\sigma^{23},\sigma^{12}\right\} + p^3p^2\left\{\sigma^{31},\sigma^{12}\right\}\Big] &(22)\\
&\frac{1}{\mathfrak{p}^2}\left[\left(p^1\right)^2 + \left(p^2\right)^2 + \left(p^3\right)^2\right] &(23)\\
&= 1 &(24)
\end{aligned}$$

QED.

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