HELICITY PROJECTION OPERATOR IN THE DIRAC EQUATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.15.

We'll revisit here the helicity operator in the Dirac equation. Helicity is defined as the component of the spin parallel to the particle's momentum. For a massive particle (that is, one travelling at less than the speed of light), the spin can be oriented at some angle other than parallel to the momentum, so the helicity is defined as the projection of the spin onto the momentum. The operator is defined as

$$\Sigma_{\mathbf{p}} \equiv \frac{\Sigma \cdot \mathbf{p}}{p} \tag{1}$$

where Σ is a 3-vector defined in terms of the spin matrices as

$$\Sigma \equiv \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right) \tag{2}$$

which amounts to the components of spin along the x, y and z axes. The quantity p is the magnitude of the 3-momentum:

$$p \equiv |\mathbf{p}| \tag{3}$$

The helicity operator can be used to create a helicity projection operator, which projects out particles with spin parallel or antiparallel to the momentum. It is defined as

$$\Pi_{\pm}(\mathbf{p}) \equiv \frac{1}{2}\left(1 \pm \Sigma_{\mathbf{p}}\right) \tag{4}$$

The helicity projection operators commute with the energy projection operators $\Lambda_{\pm}$, so it is possible to find a set of vectors that are simultaneous eigenstates of both operators. We've already seen that the states

$$u_{\pm}(\mathbf{p}) = \sqrt{E_p + m}\begin{bmatrix} \chi_{\pm} \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_{\pm} \end{bmatrix} \tag{5}$$

$$v_{\pm}(\mathbf{p}) = \pm\sqrt{E_p + m}\begin{bmatrix} \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_{\mp} \\ \chi_{\mp} \end{bmatrix} \tag{6}$$

are eigenstates of $\Lambda_{\pm}$, provided that $\chi_{\pm}$ are a pair of orthonormal 2-component vectors. Thus the problem becomes that of finding which particular set of $\chi_{\pm}$ make $u_{\pm}$ and $v_{\pm}$ eigenstates of $\Pi_{\pm}$. In the problem statement in L&P, we are given the pair

$$\chi_{+} = \frac{1}{\sqrt{2\mathfrak{p}(\mathfrak{p}+p_z)}}\begin{bmatrix} \mathfrak{p}+p_z \\ p_x+ip_y \end{bmatrix} \tag{7}$$

$$\chi_{-} = \frac{1}{\sqrt{2\mathfrak{p}(\mathfrak{p}+p_z)}}\begin{bmatrix} -p_x+ip_y \\ \mathfrak{p}+p_z \end{bmatrix} \tag{8}$$

where the factor $\frac{1}{\sqrt{2\mathfrak{p}(\mathfrak{p}+p_z)}}$ is just a normalization factor, required so that $\chi_{\pm}^{\dagger}\chi_{\pm} = 1$. Thus to show that these $\chi_{\pm}$ are eigenstates of $\Pi_{\pm}$, we can ignore the normalization factor and just show that the 2-component vectors are the required eigenstates. In what follows, we'll use the Dirac-Pauli representation of the gamma matrices, since the solutions 5 were derived using this form. Using this representation, we can calculate $\Sigma_{\mathbf{p}}$ directly using the gamma matrices, since

$$\sigma^{ij} = i\gamma^i\gamma^j \tag{9}$$

and the spatial gamma matrices are given by

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{10}$$

where the σ^i are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11}$$

Multiplying everything out, we get

$$\Sigma_{\mathbf{p}} = \frac{1}{\mathfrak{p}}\begin{bmatrix} p_z & p_x-ip_y & 0 & 0 \\ p_x+ip_y & -p_z & 0 & 0 \\ 0 & 0 & p_z & p_x-ip_y \\ 0 & 0 & p_x+ip_y & -p_z \end{bmatrix} \tag{12}$$

This gives

$$\Pi_+ = \frac{1}{2\mathfrak{p}} \begin{bmatrix} \mathfrak{p}+p_z & p_x-ip_y & 0 & 0 \\ p_x+ip_y & \mathfrak{p}-p_z & 0 & 0 \\ 0 & 0 & \mathfrak{p}+p_z & p_x-ip_y \\ 0 & 0 & p_x+ip_y & \mathfrak{p}-p_z \end{bmatrix} \tag{13}$$

$$\Pi_- = \frac{1}{2\mathfrak{p}} \begin{bmatrix} \mathfrak{p}-p_z & -p_x+ip_y & 0 & 0 \\ -p_x-ip_y & \mathfrak{p}+p_z & 0 & 0 \\ 0 & 0 & \mathfrak{p}-p_z & -p_x+ip_y \\ 0 & 0 & -p_x-ip_y & \mathfrak{p}+p_z \end{bmatrix} \tag{14}$$

Because both $\Pi_\pm$ are block diagonal matrices with the two blocks being identical 2×2 matrices, and the vectors $u_\pm$ and $v_\pm$ have the two component vectors $\chi_\pm$ in both their upper and lower 2 slots, we can work out the top-left 2×2 block of $\Pi_\pm$ multiplied by $\chi_\pm$ and thus deal with the simpler problem of multiplying 2×2 matrices instead of 4×4. If χ_+, say, is an eigenvector of the top-left 2×2 block of Π_+, then both u_+ and v_- will be eigenvectors of the full 4×4 matrix Π_+. I'll continue to use the notation $\Pi_\pm$ below but this now refers just to the top-left 2×2 block of $\Pi_\pm$ in each case. We have:

$$\begin{aligned}
\Pi_+\chi_+ &= \frac{1}{2\mathfrak{p}\sqrt{2\mathfrak{p}\,(\mathfrak{p}+p_z)}} \begin{bmatrix} \mathfrak{p}+p_z & p_x-ip_y \\ p_x+ip_y & \mathfrak{p}-p_z \end{bmatrix} \begin{bmatrix} \mathfrak{p}+p_z \\ p_x+ip_y \end{bmatrix} && (15)\\
&= \frac{1}{2\mathfrak{p}\sqrt{2\mathfrak{p}\,(\mathfrak{p}+p_z)}} \begin{bmatrix} \mathfrak{p}^2+2\mathfrak{p}p_z+p_z^2+p_x^2+p_y^2 \\ 2\mathfrak{p}\,(p_x+ip_y) \end{bmatrix} && (16)\\
&= \frac{1}{\sqrt{2\mathfrak{p}\,(\mathfrak{p}+p_z)}} \begin{bmatrix} \mathfrak{p}+p_z \\ p_x+ip_y \end{bmatrix} && (17)\\
&= \chi_+ && (18)
\end{aligned}$$

Thus χ_+ is indeed an eigenvector of the top-left 2×2 block of Π_+ and therefore $u_\pm$ is an eigenvector of the full 4×4 matrix Π_+. Doing the other calculations, we get

$$\Pi_+\chi_- = \frac{1}{2\mathfrak{p}\sqrt{2\mathfrak{p}\left(\mathfrak{p}+p_z\right)}}\begin{bmatrix} \mathfrak{p}+p_z & p_x-ip_y \\ p_x+ip_y & \mathfrak{p}-p_z \end{bmatrix}\begin{bmatrix} -p_x+ip_y \\ \mathfrak{p}+p_z \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

$$\Pi_-\chi_+ = \frac{1}{2\mathfrak{p}\sqrt{2\mathfrak{p}\left(\mathfrak{p}+p_z\right)}}\begin{bmatrix} \mathfrak{p}-p_z & -p_x+ip_y \\ -p_x-ip_y & \mathfrak{p}+p_z \end{bmatrix}\begin{bmatrix} \mathfrak{p}+p_z \\ p_x+ip_y \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

$$\Pi_-\chi_- = \frac{1}{2\mathfrak{p}\sqrt{2\mathfrak{p}\left(\mathfrak{p}+p_z\right)}}\begin{bmatrix} \mathfrak{p}-p_z & -p_x+ip_y \\ -p_x-ip_y & \mathfrak{p}+p_z \end{bmatrix}\begin{bmatrix} -p_x+ip_y \\ \mathfrak{p}+p_z \end{bmatrix} \tag{23}$$

$$= \frac{1}{\sqrt{2\mathfrak{p}\left(\mathfrak{p}+p_z\right)}}\begin{bmatrix} -p_x+ip_y \\ \mathfrak{p}+p_z \end{bmatrix} \tag{24}$$

$$= \chi_- \tag{25}$$

Comparing with 5, we see that u_+ and v_- are eigenvectors of Π_+ with eigenvalue $+1$ and of Π_- with eigenvalue 0, u_- and v_+ are eigenvectors of Π_- with eigenvalue 1, and of Π_+ with eigenvalue 0.

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