HELICITY PROJECTION OPERATOR: PROPERTIES AND COMMUTATORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problems 4.16 - 4.17.

We've seen that the helicity projection operator is given by

$$\Pi_{\pm}(\mathbf{p}) \equiv \frac{1}{2}(1 \pm \Sigma_{\mathbf{p}}) \tag{1}$$

where

$$\Sigma_{\mathbf{p}} \equiv \frac{\Sigma \cdot \mathbf{p}}{p} \tag{2}$$

where Σ is a 3-vector defined in terms of the spin matrices as

$$\Sigma \equiv \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right) \tag{3}$$

which amounts to the components of spin along the x, y and z axes. The quantity p is the magnitude of the 3-momentum:

$$p \equiv |\mathbf{p}| \tag{4}$$

We can now derive a few properties of this operator. First, since we know that

$$\Sigma_{\mathbf{p}}^2 = I \tag{5}$$

we have

$$\Pi_{\pm}^2 = \frac{1}{4}\left(1 \pm 2\Sigma_{\mathbf{p}} + \Sigma_{\mathbf{p}}^2\right) \tag{6}$$

$$= \frac{1}{2}(1 \pm \Sigma_{\mathbf{p}}) \tag{7}$$

$$= \Pi_{\pm} \tag{8}$$

Also

$$\Pi_+\Pi_- = \frac{1}{4}(1+\Sigma_{\mathbf{p}})(1-\Sigma_{\mathbf{p}}) \tag{9}$$
$$= \frac{1}{4}\left(1-\Sigma_{\mathbf{p}}^2\right) \tag{10}$$
$$= 0 \tag{11}$$
$$\Pi_-\Pi_+ = \frac{1}{4}(1-\Sigma_{\mathbf{p}})(1+\Sigma_{\mathbf{p}}) \tag{12}$$
$$= \frac{1}{4}\left(1-\Sigma_{\mathbf{p}}^2\right) \tag{13}$$
$$= 0 \tag{14}$$

Finally, it's fairly obvious that

$$\Pi_+ + \Pi_- = 1 \tag{15}$$

We can also consider the commutators of $\Pi_\pm$ with the energy projection operators $\Lambda_\pm$. We'll look at $[\Lambda_+, \Pi_\pm]$ since the other commutator $[\Lambda_+, \Pi_\pm]$ is similar. From its definition

$$\Lambda_+ = \frac{\not{p}+m}{2m} \tag{16}$$
$$= \frac{1}{2m}\left(\gamma^\mu p_\mu + m\right) \tag{17}$$

To find $[\Lambda_+, \Pi_\pm]$, we need consider only $\left[\gamma^\mu p_\mu, \Sigma_{\mathbf{p}}\right]$ since the m in Λ_+ is just a multiple of the unit matrix, and the other component of Π_+ is the unit operator which commutes with everything. We can write out $\Sigma_{\mathbf{p}}$ in terms of the σ^{ij} so we have

$$\Sigma_{\mathbf{p}} = \frac{1}{\not{\mathbf{p}}}\left(\sigma^{23}p^1 + \sigma^{31}p^2 + \sigma^{12}p^3\right) \tag{18}$$

We therefore need the various commutators $\left[\gamma^\mu, \sigma^{ij}\right]$, which we've already worked out

$$\left[\gamma^\mu, \sigma^{ij}\right] = 2i\left(g^{\mu i}\gamma^j - g^{\mu j}\gamma^i\right) \tag{19}$$

Plugging this into the commutator, we have

$$\left[\gamma^{\mu} p_{\mu}, \Sigma_{\mathbf{p}}\right] = \frac{2i}{2m\mathfrak{p}} \left[\left(g^{\mu 2}\gamma^{3} - g^{\mu 3}\gamma^{2}\right) p_{\mu} p^{1} + \left(g^{\mu 3}\gamma^{1} - g^{\mu 1}\gamma^{3}\right) p_{\mu} p^{2} + \right. \tag{20}$$

$$\left. \left(g^{\mu 1}\gamma^{2} - g^{\mu 2}\gamma^{1}\right) p_{\mu} p^{3}\right] \tag{21}$$

$$= \frac{i}{m\mathfrak{p}} \left[\gamma^{3} p^{2} p^{1} - \gamma^{2} p^{3} p^{1} + \gamma^{1} p^{3} p^{2} - \gamma^{3} p^{1} p^{2} + \gamma^{2} p^{1} p^{3} - \gamma^{1} p^{2} p^{3}\right] \tag{22}$$

$$= 0 \tag{23}$$

Therefore

$$[\Lambda_{+}, \Pi_{\pm}] = 0 \tag{24}$$

The other commutator, $[\Lambda_{-}, \Pi_{\pm}]$, just requires replacing p_{μ} by $-p_{\mu}$ in 20 so is also zero, so we have

$$[\Lambda_{+}, \Pi_{\pm}] = [\Lambda_{-}, \Pi_{\pm}] = 0 \tag{25}$$