

HELICITY PROJECTION OPERATOR: PROPERTIES AND COMMUTATORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problems 4.16 - 4.17.

We've seen that the helicity projection operator is given by

$$\Pi_{\pm}(\mathbf{p}) \equiv \frac{1}{2}(1 \pm \Sigma_{\mathbf{p}}) \quad (1)$$

where

$$\Sigma_{\mathbf{p}} \equiv \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p} \quad (2)$$

where $\boldsymbol{\Sigma}$ is a 3-vector defined in terms of the spin matrices as

$$\boldsymbol{\Sigma} \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12}) \quad (3)$$

which amounts to the components of spin along the x , y and z axes. The quantity p is the magnitude of the 3-momentum:

$$p \equiv |\mathbf{p}| \quad (4)$$

We can now derive a few properties of this operator. First, since we know that

$$\Sigma_{\mathbf{p}}^2 = I \quad (5)$$

we have

$$\Pi_{\pm}^2 = \frac{1}{4}(1 \pm 2\Sigma_{\mathbf{p}} + \Sigma_{\mathbf{p}}^2) \quad (6)$$

$$= \frac{1}{2}(1 \pm \Sigma_{\mathbf{p}}) \quad (7)$$

$$= \Pi_{\pm} \quad (8)$$

Also

$$\Pi_+ \Pi_- = \frac{1}{4} (1 + \Sigma_{\mathbf{p}}) (1 - \Sigma_{\mathbf{p}}) \quad (9)$$

$$= \frac{1}{4} (1 - \Sigma_{\mathbf{p}}^2) \quad (10)$$

$$= 0 \quad (11)$$

$$\Pi_- \Pi_+ = \frac{1}{4} (1 - \Sigma_{\mathbf{p}}) (1 + \Sigma_{\mathbf{p}}) \quad (12)$$

$$= \frac{1}{4} (1 - \Sigma_{\mathbf{p}}^2) \quad (13)$$

$$= 0 \quad (14)$$

Finally, it's fairly obvious that

$$\Pi_+ + \Pi_- = 1 \quad (15)$$

We can also consider the commutators of Π_{\pm} with the energy projection operators Λ_{\pm} . We'll look at $[\Lambda_+, \Pi_{\pm}]$ since the other commutator $[\Lambda_-, \Pi_{\pm}]$ is similar. From its definition

$$\Lambda_+ = \frac{\not{p} + m}{2m} \quad (16)$$

$$= \frac{1}{2m} (\gamma^{\mu} p_{\mu} + m) \quad (17)$$

To find $[\Lambda_+, \Pi_{\pm}]$, we need consider only $[\gamma^{\mu} p_{\mu}, \Sigma_{\mathbf{p}}]$ since the m in Λ_+ is just a multiple of the unit matrix, and the other component of Π_+ is the unit operator which commutes with everything. We can write out $\Sigma_{\mathbf{p}}$ in terms of the σ^{ij} so we have

$$\Sigma_{\mathbf{p}} = \frac{1}{\mathbf{p}} (\sigma^{23} p^1 + \sigma^{31} p^2 + \sigma^{12} p^3) \quad (18)$$

We therefore need the various commutators $[\gamma^{\mu}, \sigma^{ij}]$, which we've already worked out

$$[\gamma^{\mu}, \sigma^{ij}] = 2i (g^{\mu i} \gamma^j - g^{\mu j} \gamma^i) \quad (19)$$

Plugging this into the commutator, we have

$$[\gamma^\mu p_\mu, \Sigma_{\mathbf{p}}] = \frac{2i}{2m_{\mathbf{p}}} [(g^{\mu 2} \gamma^3 - g^{\mu 3} \gamma^2) p_\mu p^1 + (g^{\mu 3} \gamma^1 - g^{\mu 1} \gamma^3) p_\mu p^2 + \quad (20)$$

$$(g^{\mu 1} \gamma^2 - g^{\mu 2} \gamma^1) p_\mu p^3] \quad (21)$$

$$= \frac{i}{m_{\mathbf{p}}} [\gamma^3 p^2 p^1 - \gamma^2 p^3 p^1 + \gamma^1 p^3 p^2 - \gamma^3 p^1 p^2 + \gamma^2 p^1 p^3 - \gamma^1 p^2 p^3] \quad (22)$$

$$= 0 \quad (23)$$

Therefore

$$[\Lambda_+, \Pi_\pm] = 0 \quad (24)$$

The other commutator, $[\Lambda_-, \Pi_\pm]$, just requires replacing p_μ by $-p_\mu$ in 20 so is also zero, so we have

$$[\Lambda_+, \Pi_\pm] = [\Lambda_-, \Pi_\pm] = 0 \quad (25)$$