CHIRALITY PROJECTION OPERATOR AND EIGENVECTORS OF γ^5

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Post date: 17 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.18.

In general, any operator Q whose square is the identity matrix, so that $Q^2 = I$ can be used to define a projection operator P according to

$$P_{\pm} = \frac{1}{2}(I \pm Q) \tag{1}$$

We can see by direct calculation that

$$P_{\pm}^2 = P_{\pm} \tag{2}$$
$$P_+P_- = P_-P_+ = 0 \tag{3}$$
$$P_+ + P_- = I \tag{4}$$

Thus $P_{\pm}$ project an arbitrary state into two substates, determined by some property that can take on one of two values, such as positive or negative energy, or helicity. The matrix γ^5 satisfies $\left(\gamma^5\right)^2 = 1$, so can be used to create a pair of projection operators known as the *chirality* operators:

$$\mathsf{L} = \frac{1}{2}(1 - \gamma_5) \tag{5}$$
$$\mathsf{R} = \frac{1}{2}(1 + \gamma_5) \tag{6}$$

One property of γ_5 is that its eigenvectors cannot be written as linear combinations of the $u_{\pm}$ spinors defined earlier:

$$u_{\pm}(\mathbf{p}) = \sqrt{E_p + m}\begin{bmatrix} \chi_{\pm} \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_{\pm} \end{bmatrix} \tag{7}$$

where $\chi_{\pm}$ are a pair of orthonormal 2-component vectors. The easiest way to see this is to work in the Dirac-Pauli representation in which

$$\gamma^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{8}$$

By direct calculation (I used Maple, although the actual calculation isn't too difficult if done by hand), we find that γ^5 has 2 distinct eigenvalues of ± 1, with the eigenvectors of each eigenvalue spanning a 2-dimensional subspace. The eigenvalues and their corresponding eigenvectors are

$$\lambda = -1; \quad \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{9}$$

$$\lambda = +1; \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{10}$$

Thus the most general eigenstates of γ^5 are

$$A = a \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -b \\ -a \\ b \\ a \end{bmatrix} \tag{11}$$

$$B = c \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} d \\ c \\ d \\ c \end{bmatrix} \tag{12}$$

Let's compare these with 7, where to simplify the notation we'll define

$$\alpha \equiv \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \tag{13}$$

and ignore the normalization factor of $\sqrt{E_p + m}$, since it doesn't affect the eigenvector equations. If we try to form a linear combination of u_+ and u_- that is an eigenstate of γ^5 with eigenvalue -1, then we must have

$$eu_+ + fu_- = A = \begin{bmatrix} -b \\ -a \\ b \\ a \end{bmatrix} \tag{14}$$

which gives us

$$e\chi_{+1} + f\chi_{-1} = -b \tag{15}$$
$$e\chi_{+2} + f\chi_{-2} = -a \tag{16}$$
$$e\chi_{+1} + f\chi_{-1} = \frac{b}{\alpha} \tag{17}$$
$$e\chi_{+2} + f\chi_{-2} = \frac{a}{\alpha} \tag{18}$$

Thus we must have

$$\alpha = -1 \tag{19}$$

in order to satisfy these four equations. From 13, this can happen only if $m = 0$ and the spin is antiparallel to the momentum, so that $E_p = -|\boldsymbol{\sigma}\cdot\mathbf{p}|$.

If we try to form a linear combination of u_+ and u_- that is an eigenstate of γ^5 with eigenvalue $+1$, we find that this is possible only if $m = 0$ and $E_p = +|\boldsymbol{\sigma}\cdot\mathbf{p}|$.

If the particle is massless, then

$$u_+(\mathbf{p}) = \mathfrak{p}\begin{bmatrix} \chi_+ \\ \pm\chi_+ \end{bmatrix} \tag{20}$$
$$u_-(\mathbf{p}) = \mathfrak{p}\begin{bmatrix} \chi_- \\ \pm\chi_- \end{bmatrix} \tag{21}$$

which are (from L&P's equation 4.77) eigenstates of the helicity operator Σ_p. However, these are also eigenstates of γ^5. For example, for u_+ we have

$$\begin{bmatrix} \chi_+ \\ \chi_+ \end{bmatrix} = \chi_{+2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \chi_{+1}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{22}$$

which is an eigenstate of γ^5 with eigenvalue $+1$;

$$\begin{bmatrix} \chi_+ \\ -\chi_+ \end{bmatrix} = -\chi_{+2}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \chi_{+1}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{23}$$

which is an eigenstate of γ^5 with eigenvalue -1. The state u_- follows the same pattern, but with χ_+ replaced by χ_-.

PINGBACKS

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