

## HELICITY AND CHIRALITY ARE EQUAL FOR MASSLESS FERMIONS

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Post date: 17 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.19.

In the Dirac equation, the helicity projection operators are defined as

$$\Pi_{\pm}(\mathbf{p}) \equiv \frac{1}{2}(1 \pm \Sigma_{\mathbf{p}}) \quad (1)$$

where

$$\Sigma_{\mathbf{p}} \equiv \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p} \quad (2)$$

and  $\boldsymbol{\Sigma}$  is a 3-vector defined in terms of the spin matrices as

$$\boldsymbol{\Sigma} \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12}) \quad (3)$$

which amounts to the components of spin along the  $x$ ,  $y$  and  $z$  axes. The quantity  $p$  is the magnitude of the 3-momentum:

$$p \equiv |\mathbf{p}| \quad (4)$$

The chirality projection operators are

$$L = \frac{1}{2}(1 - \gamma_5) \quad (5)$$

$$R = \frac{1}{2}(1 + \gamma_5) \quad (6)$$

We can see that these two operators are in fact the same for massless fermions, and we can do this generically, without using any particular representation of the  $\gamma^{\mu}$ .

If  $m = 0$ , the Dirac equation is

$$i\gamma^{\mu}\partial_{\mu}\psi = 0 \quad (7)$$

The positive energy solution of this equation is

$$\psi = u(\mathbf{p}) e^{-ip \cdot x} \quad (8)$$

We can rearrange this to get

$$(\gamma^0 p_0 - \boldsymbol{\gamma} \cdot \mathbf{p}) u = 0 \quad (9)$$

$$\gamma^0 p_0 u = \boldsymbol{\gamma} \cdot \mathbf{p} u \quad (10)$$

$$\gamma^5 \gamma^0 \gamma^0 p_0 u = \gamma^5 \boldsymbol{\gamma} \cdot \mathbf{p} u \quad (11)$$

$$p_0 \gamma^5 u = \gamma^5 \gamma^0 (\gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3) u \quad (12)$$

In the third line, we multiplied both sides from the left by  $\gamma_5 \gamma^0$ .

Using the definition of  $\gamma_5$

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad (13)$$

and the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (14)$$

and the squares

$$(\gamma^0)^2 = 1 \quad (15)$$

$$(\gamma^i)^2 = -1 \quad (16)$$

we have, for the first term on the RHS of 12:

$$\gamma^5 \gamma^0 \gamma^1 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^1 \quad (17)$$

$$= (-1)^3 i\gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 \quad (18)$$

$$= -i\gamma^1 \gamma^2 \gamma^3 \gamma^1 \quad (19)$$

$$= -(-1)^2 i\gamma^1 \gamma^1 \gamma^2 \gamma^3 \quad (20)$$

$$= i\gamma^2 \gamma^3 \quad (21)$$

$$= \sigma^{23} \quad (22)$$

using the definition of the  $\sigma^{ij}$ s. Using the same technique, we find for the other two terms:

$$\gamma^5 \gamma^0 \gamma^2 = \sigma^{31} \quad (23)$$

$$\gamma^5 \gamma^0 \gamma^3 = \sigma^{12} \quad (24)$$

Comparing this with 3 we see that the RHS of 12 is

$$\gamma^5 \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{p} u = \boldsymbol{\Sigma} \cdot \mathbf{p} u \quad (25)$$

Since for a massless particle

$$p_0 = E_p = \mathbf{p} \quad (26)$$

we have, from 25 and 12

$$\gamma^5 u = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{\mathbf{p}} u \quad (27)$$

or, in terms of projection operators

$$\frac{1}{2} \left( 1 \pm \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{\mathbf{p}} \right) u = \Pi_{\pm} u = \frac{1}{2} \left( 1 \pm \gamma^5 \right) u \quad (28)$$

Thus in the limit of zero mass, for fermions:

$$\Pi_{\pm} \rightarrow \frac{1}{2} \left( 1 \pm \gamma^5 \right) \quad (29)$$