HELICITY AND CHIRALITY ARE EQUAL FOR MASSLESS FERMIONS

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Post date: 17 June 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.19.

In the Dirac equation, the helicity projection operators are defined as

$$\Pi_{\pm}(\mathbf{p}) \equiv \frac{1}{2}(1 \pm \Sigma_{\mathbf{p}}) \tag{1}$$

where

$$\Sigma_{\mathbf{p}} \equiv \frac{\Sigma \cdot \mathbf{p}}{\mathbf{p}} \tag{2}$$

and Σ is a 3-vector defined in terms of the spin matrices as

$$\Sigma \equiv \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right) \tag{3}$$

which amounts to the components of spin along the x, y and z axes. The quantity p is the magnitude of the 3-momentum:

$$p \equiv |\mathbf{p}| \tag{4}$$

The chirality projection operators are

$$\mathsf{L} = \frac{1}{2}(1 - \gamma_5) \tag{5}$$

$$\mathsf{R} = \frac{1}{2}(1 + \gamma_5) \tag{6}$$

We can see that these two operators are in fact the same for massless fermions, and we can do this generically, without using any particular representation of the γ^{μ}.

If $m = 0$, the Dirac equation is

$$i\gamma^{\mu}\partial_{\mu}\psi = 0 \tag{7}$$

The positive energy solution of this equation is

$$\psi = u(\mathbf{p})\, e^{-ip\cdot x} \tag{8}$$

We can rearrange this to get

$$\left(\gamma^0 p_0 - \gamma\cdot\mathbf{p}\right) u = 0 \tag{9}$$
$$\gamma^0 p_0 u = \gamma\cdot\mathbf{p}u \tag{10}$$
$$\gamma^5\gamma^0\gamma^0 p_0 u = \gamma^5\gamma^0\gamma\cdot\mathbf{p}u \tag{11}$$
$$p_0\gamma^5 u = \gamma^5\gamma^0\left(\gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3\right) u \tag{12}$$

In the third line, we multiplied both sides from the left by $\gamma_5\gamma^0$.

Using the definition of γ_5

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{13}$$

and the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{14}$$

and the squares

$$\left(\gamma^0\right)^2 = 1 \tag{15}$$
$$\left(\gamma^i\right)^2 = -1 \tag{16}$$

we have, for the first term on the RHS of 12:

$$\gamma^5\gamma^0\gamma^1 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1 \tag{17}$$
$$= (-1)^3\, i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 \tag{18}$$
$$= -i\gamma^1\gamma^2\gamma^3\gamma^1 \tag{19}$$
$$= -(-1)^2\, i\gamma^1\gamma^1\gamma^2\gamma^3 \tag{20}$$
$$= i\gamma^2\gamma^3 \tag{21}$$
$$= \sigma^{23} \tag{22}$$

using the definition of the σ^{ij}s. Using the same technique, we find for the other two terms:

$$\gamma^5\gamma^0\gamma^2 = \sigma^{31} \tag{23}$$
$$\gamma^5\gamma^0\gamma^3 = \sigma^{12} \tag{24}$$

Comparing this with 3 we see that the RHS of 12 is

$$\gamma^5\gamma^0\gamma\cdot\mathbf{p}u = \Sigma\cdot\mathbf{p} \tag{25}$$

Since for a massless particle

$$p_0 = E_p = \mathfrak{p} \tag{26}$$

we have, from 25 and 12

$$\gamma^5 u = \frac{\Sigma\cdot\mathbf{p}}{\mathfrak{p}}u \tag{27}$$

or, in terms of projection operators

$$\frac{1}{2}\left(1 \pm \frac{\Sigma\cdot\mathbf{p}}{\mathfrak{p}}\right)u = \Pi_\pm u = \frac{1}{2}\left(1 \pm \gamma^5\right)u \tag{28}$$

Thus in the limit of zero mass, for fermions:

$$\Pi_\pm \to \frac{1}{2}\left(1 \pm \gamma^5\right) \tag{29}$$