SPIN PROJECTION OPERATORS IN THE DIRAC EQUATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.20.

In the Dirac equation, the helicity projection operators project the component of spin along the direction of 3-momentum. Since this requires that the particle actually has a non-zero 3-momentum, so they cannot be used to project spin components for a particle at rest. For this we can define a pair of new projection operators called the *spin projection operators*.

To do this, we first define the 4-vector n^{μ} with components

$$n^0 = \frac{\mathbf{p}\cdot\hat{\mathbf{s}}}{m} \tag{1}$$

$$\mathbf{n} = \hat{\mathbf{s}} + \frac{(\mathbf{p}\cdot\hat{\mathbf{s}})\,\mathbf{p}}{m(E+m)} \tag{2}$$

where $\hat{\mathbf{s}}$ denotes a unit vector in a particular spatial direction and $\mathbf{p}$ is the usual 3-momentum and $E=\sqrt{\mathbf{p}^2+m^2}$. As a four vector

$$p^{\mu} = (E,\mathbf{p}) \tag{3}$$

We first note a couple of properties of n^{μ}.

First, we have

$$\begin{aligned}
p^{\mu}n_{\mu} &= \frac{E}{m}\mathbf{p}\cdot\hat{\mathbf{s}} - \mathbf{p}\cdot\mathbf{n} && (4)\\
&= \frac{E}{m}\mathbf{p}\cdot\hat{\mathbf{s}} - \mathbf{p}\cdot\hat{\mathbf{s}} - \frac{\mathbf{p}\cdot\hat{\mathbf{s}}}{m}\frac{\mathbf{p}^2}{(E+m)} && (5)\\
&= \frac{\mathbf{p}\cdot\hat{\mathbf{s}}}{m(E+m)}\left[E(E+m) - m(E+m) - \mathbf{p}^2\right] && (6)\\
&= \frac{\mathbf{p}\cdot\hat{\mathbf{s}}}{m(E+m)}\left[E^2 - m^2 - \mathbf{p}^2\right] && (7)\\
&= 0 && (8)
\end{aligned}$$

Next, we have, using $\hat{\mathbf{s}}^2=1$ since it's a unit vector:

$$n^{\mu}n_{\mu}=\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})^{2}}{m^{2}}-\left(\hat{\mathbf{s}}+\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})\,\mathbf{p}}{m\,(E+m)}\right)^{2} \tag{9}$$

$$=\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})^{2}}{m^{2}}-\hat{\mathbf{s}}^{2}-2\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})^{2}}{m\,(E+m)}-\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})^{2}\,\mathbf{p}^{2}}{m^{2}\,(E+m)^{2}} \tag{10}$$

$$=-1+\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})^{2}}{m^{2}\,(E+m)^{2}}\left[(E+m)^{2}-2m\,(E+m)-\mathbf{p}^{2}\right] \tag{11}$$

$$=-1+\frac{(\mathbf{p}\cdot\hat{\mathbf{s}})^{2}}{m^{2}\,(E+m)^{2}}\left[E^{2}-m^{2}-\mathbf{p}^{2}\right] \tag{12}$$

$$=-1 \tag{13}$$

Finally, we note that, using the properties of the γ^{μ} and γ_{5}, and the identity $\not{n}\not{n}=n^{\mu}n_{\mu}$:

$$(\gamma_{5}\not{n})^{2}=\gamma_{5}\gamma^{\mu}n_{\mu}\gamma_{5}\gamma^{\nu}n_{\nu} \tag{14}$$

$$=-\gamma^{\mu}n_{\mu}\,(\gamma_{5})^{2}\,\gamma^{\nu}n_{\nu} \tag{15}$$

$$=-\not{n}\not{n} \tag{16}$$

$$=-n^{\mu}n_{\mu} \tag{17}$$

$$=+1 \tag{18}$$

Because $(\gamma_{5}\not{n})^{2}=1$, we can use it to create a pair of projection operators, which we define as

$$P_{\uparrow}\equiv\frac{1}{2}\,(1+\gamma_{5}\not{n}) \tag{19}$$

$$P_{\downarrow}\equiv\frac{1}{2}\,(1-\gamma_{5}\not{n}) \tag{20}$$

To see the meaning of these projection operators, we'll use the example in L&P, with

$$\hat{\mathbf{s}}=(1,0,0) \tag{21}$$

so we're looking in the x direction. In the particle's rest frame, from 2 we have

$$n^{\mu}=(0,\hat{\mathbf{s}})=(0,1,0,0) \tag{22}$$

so in this case

$$\not{n}=\gamma_{1} \tag{23}$$

This gives

$$\gamma_5 \not{n} = \gamma_5 \gamma_1 \tag{24}$$
$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma_1 \tag{25}$$
$$= i\gamma^0\gamma^1\gamma_1\gamma^2\gamma^3 \tag{26}$$
$$= i\gamma^0 g_{11}\left(\gamma^1\right)^2\gamma^2\gamma^3 \tag{27}$$
$$= i\gamma^0\gamma^2\gamma^3 \tag{28}$$
$$= \gamma^0\sigma^{23} \tag{29}$$

In the Dirac-Pauli representation of the γ^μ we have

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{30}$$
$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \tag{31}$$

so multiplying out $\gamma^2\gamma^3$ we have

$$\sigma^{23} = \begin{bmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{bmatrix} \tag{32}$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{33}$$
$$\gamma^0\sigma^{23} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{34}$$
$$P_\uparrow = \frac{1}{2}\left(I + \gamma^0\sigma^{23}\right) \tag{35}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{36}$$

In the particle's rest frame, we've seen that the following spinors are solutions

$$u_{+}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \tag{37}$$

$$u_{-}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \tag{38}$$

$$v_{+}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{39}$$

$$v_{-}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}0\\0\\-1\\0\end{bmatrix} \tag{40}$$

We can see the effects of acting on these states with the spin projection operator $P_{\uparrow}$:

$$P_{\uparrow}u_{+}(\mathbf{0})=\frac{\sqrt{2m}}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \tag{41}$$

The operator 32 is the x component of the spin operator Σ_x, and if we operate on this state with Σ_x we have

$$\Sigma_x P_{\uparrow}u_{+}(\mathbf{0})=\frac{\sqrt{2m}}{2}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \tag{42}$$

$$=\frac{\sqrt{2m}}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \tag{43}$$

$$=P_{\uparrow}u_{+}(\mathbf{0}) \tag{44}$$

Thus $P_{\uparrow}u_{+}(\mathbf{0})$ is an eigenstate of Σ_x with eigenvalue $+1$. We can do similar calculations for the other three spinors to find that

$$\Sigma_x P_{\uparrow} u_{\pm}(\mathbf{0}) = P_{\uparrow} u_{\pm}(\mathbf{0}) \tag{45}$$

$$\Sigma_x P_{\uparrow} v_{\pm}(\mathbf{0}) = -P_{\uparrow} v_{\pm}(\mathbf{0}) \tag{46}$$

Thus the states $u_{\pm}$ and $v_{\pm}$ have well-defined spin components in the x, which are projected out by the operator $P_{\uparrow}$.

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