

SPIN PROJECTION OPERATORS IN THE DIRAC EQUATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.20.

In the Dirac equation, the helicity projection operators project the component of spin along the direction of 3-momentum. Since this requires that the particle actually has a non-zero 3-momentum, so they cannot be used to project spin components for a particle at rest. For this we can define a pair of new projection operators called the *spin projection operators*.

To do this, we first define the 4-vector n^μ with components

$$n^0 = \frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m} \quad (1)$$

$$\mathbf{n} = \hat{\mathbf{s}} + \frac{(\mathbf{p} \cdot \hat{\mathbf{s}}) \mathbf{p}}{m(E + m)} \quad (2)$$

where $\hat{\mathbf{s}}$ denotes a unit vector in a particular spatial direction and \mathbf{p} is the usual 3-momentum and $E = \sqrt{\mathbf{p}^2 + m^2}$. As a four vector

$$p^\mu = (E, \mathbf{p}) \quad (3)$$

We first note a couple of properties of n^μ .

First, we have

$$p^\mu n_\mu = \frac{E}{m} \mathbf{p} \cdot \hat{\mathbf{s}} - \mathbf{p} \cdot \mathbf{n} \quad (4)$$

$$= \frac{E}{m} \mathbf{p} \cdot \hat{\mathbf{s}} - \mathbf{p} \cdot \hat{\mathbf{s}} - \frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m} \frac{\mathbf{p}^2}{(E + m)} \quad (5)$$

$$= \frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m(E + m)} [E(E + m) - m(E + m) - \mathbf{p}^2] \quad (6)$$

$$= \frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m(E + m)} [E^2 - m^2 - \mathbf{p}^2] \quad (7)$$

$$= 0 \quad (8)$$

Next, we have, using $\hat{\mathbf{s}}^2 = 1$ since it's a unit vector:

$$n^\mu n_\mu = \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})^2}{m^2} - \left(\hat{\mathbf{s}} + \frac{(\mathbf{p} \cdot \hat{\mathbf{s}}) \mathbf{p}}{m(E+m)} \right)^2 \quad (9)$$

$$= \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})^2}{m^2} - \hat{\mathbf{s}}^2 - 2 \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})^2}{m(E+m)} - \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})^2 \mathbf{p}^2}{m^2(E+m)^2} \quad (10)$$

$$= -1 + \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})^2}{m^2(E+m)^2} \left[(E+m)^2 - 2m(E+m) - \mathbf{p}^2 \right] \quad (11)$$

$$= -1 + \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})^2}{m^2(E+m)^2} [E^2 - m^2 - \mathbf{p}^2] \quad (12)$$

$$= -1 \quad (13)$$

Finally, we note that, using the properties of the γ^μ and γ_5 , and the identity $\not{n} \not{n} = n^\mu n_\mu$:

$$(\gamma_5 \not{n})^2 = \gamma_5 \gamma^\mu n_\mu \gamma_5 \gamma^\nu n_\nu \quad (14)$$

$$= -\gamma^\mu n_\mu (\gamma_5)^2 \gamma^\nu n_\nu \quad (15)$$

$$= -\not{n} \not{n} \quad (16)$$

$$= -n^\mu n_\mu \quad (17)$$

$$= +1 \quad (18)$$

Because $(\gamma_5 \not{n})^2 = 1$, we can use it to create a pair of projection operators, which we define as

$$P_\uparrow \equiv \frac{1}{2} (1 + \gamma_5 \not{n}) \quad (19)$$

$$P_\downarrow \equiv \frac{1}{2} (1 - \gamma_5 \not{n}) \quad (20)$$

To see the meaning of these projection operators, we'll use the example in L&P, with

$$\hat{\mathbf{s}} = (1, 0, 0) \quad (21)$$

so we're looking in the x direction. In the particle's rest frame, from 2 we have

$$n^\mu = (0, \hat{\mathbf{s}}) = (0, 1, 0, 0) \quad (22)$$

so in this case

$$\not{n} = \gamma_1 \quad (23)$$

This gives

$$\gamma_5 \not{t} = \gamma_5 \gamma_1 \quad (24)$$

$$= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma_1 \quad (25)$$

$$= i\gamma^0 \gamma^1 \gamma_1 \gamma^2 \gamma^3 \quad (26)$$

$$= i\gamma^0 g_{11} (\gamma^1)^2 \gamma^2 \gamma^3 \quad (27)$$

$$= i\gamma^0 \gamma^2 \gamma^3 \quad (28)$$

$$= \gamma^0 \sigma^{23} \quad (29)$$

In the Dirac-Pauli representation of the γ^μ we have

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (30)$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad (31)$$

so multiplying out $\gamma^2 \gamma^3$ we have

$$\sigma^{23} = \begin{bmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{bmatrix} \quad (32)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (33)$$

$$\gamma^0 \sigma^{23} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (34)$$

$$P_\uparrow = \frac{1}{2} (I + \gamma^0 \sigma^{23}) \quad (35)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad (36)$$

In the particle's rest frame, we've seen that the following spinors are solutions

$$u_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (37)$$

$$u_-(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (38)$$

$$v_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (39)$$

$$v_-(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad (40)$$

We can see the effects of acting on these states with the spin projection operator P_{\uparrow} :

$$P_{\uparrow}u_+(\mathbf{0}) = \frac{\sqrt{2m}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (41)$$

The operator 32 is the x component of the spin operator Σ_x , and if we operate on this state with Σ_x we have

$$\Sigma_x P_{\uparrow}u_+(\mathbf{0}) = \frac{\sqrt{2m}}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (42)$$

$$= \frac{\sqrt{2m}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (43)$$

$$= P_{\uparrow}u_+(\mathbf{0}) \quad (44)$$

Thus $P_{\uparrow}u_+(\mathbf{0})$ is an eigenstate of Σ_x with eigenvalue $+1$. We can do similar calculations for the other three spinors to find that

$$\Sigma_x P_{\uparrow} u_{\pm}(\mathbf{0}) = P_{\uparrow} u_{\pm}(\mathbf{0}) \quad (45)$$

$$\Sigma_x P_{\uparrow} v_{\pm}(\mathbf{0}) = -P_{\uparrow} v_{\pm}(\mathbf{0}) \quad (46)$$

Thus the states u_{\pm} and v_{\pm} have well-defined spin components in the x , which are projected out by the operator P_{\uparrow} .

PINGBACKS

Pingback: Spin projection operators in the Dirac equation for a particle at rest