

## SPIN PROJECTION OPERATORS IN THE DIRAC EQUATION FOR A PARTICLE AT REST

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.21.

In the last post, we saw that the spin projection operators projected out spin components for a particle at rest, but we used the explicit Dirac-Pauli representation of the  $\gamma^\mu$ . It is possible to show the same thing generally, without using any particular representation for the  $\gamma^\mu$ . That is, we want to show that for the spinors representing a particle at rest, given by

$$u_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

$$u_-(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

$$v_+(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (3)$$

$$v_-(\mathbf{0}) = \sqrt{2m} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad (4)$$

and a projection operator given by

$$P_\uparrow = \frac{1}{2} (I + \gamma^0 \sigma^{23}) \quad (5)$$

we get

$$\Sigma_x P_\uparrow u_\pm(\mathbf{0}) = P_\uparrow u_\pm(\mathbf{0}) \quad (6)$$

$$\Sigma_x P_\uparrow v_\pm(\mathbf{0}) = -P_\uparrow v_\pm(\mathbf{0}) \quad (7)$$

To do this, we first note that  $u_\pm(\mathbf{0})$  and  $v_\pm(\mathbf{0})$  are eigenstates of  $\gamma^0$ . From the definition of these spinors, they satisfy the equations

$$(\not{p} - m) u_\pm(\mathbf{p}) = 0 \quad (8)$$

$$(\not{p} + m) v_\pm(\mathbf{p}) = 0 \quad (9)$$

With  $\mathbf{p} = 0$ ,  $\not{p} = \gamma^0 p_0 = m\gamma^0$  so we have

$$(\not{p} - m) u_\pm(\mathbf{0}) = (m\gamma^0 - m) u_\pm(\mathbf{0}) = 0 \quad (10)$$

so

$$\gamma^0 u_\pm(\mathbf{0}) = u_\pm(\mathbf{0}) \quad (11)$$

Similarly

$$\gamma^0 v_\pm(\mathbf{0}) = -v_\pm(\mathbf{0}) \quad (12)$$

Returning to 5, we have

$$P_\uparrow u_\pm(\mathbf{0}) = \frac{1}{2} (I + \gamma^0 \sigma^{23}) u_\pm(\mathbf{0}) \quad (13)$$

$$= \frac{1}{2} (I + i\gamma^0 \gamma^2 \gamma^3) u_\pm(\mathbf{0}) \quad (14)$$

$$= \frac{1}{2} (I + i\gamma^2 \gamma^3 \gamma^0) u_\pm(\mathbf{0}) \quad (15)$$

$$= \frac{1}{2} (I + i\gamma^2 \gamma^3) u_\pm(\mathbf{0}) \quad (16)$$

$$= \frac{1}{2} (I + \Sigma_x) u_\pm(\mathbf{0}) \quad (17)$$

We can now operate on this state with  $\Sigma_x = i\gamma^2 \gamma^3$  to get

$$\Sigma_x P_{\uparrow} u_{\pm}(\mathbf{0}) = \Sigma_x \frac{1}{2} (I + \Sigma_x) u_{\pm}(\mathbf{0}) \quad (18)$$

$$= \frac{1}{2} \left( \Sigma_x + (\Sigma_x)^2 \right) u_{\pm}(\mathbf{0}) \quad (19)$$

$$= \frac{1}{2} (\Sigma_x + I) u_{\pm}(\mathbf{0}) \quad (20)$$

$$= P_{\uparrow} u_{\pm}(\mathbf{0}) \quad (21)$$

where we've used

$$(\Sigma_x)^2 = (i\gamma^2 \gamma^3)^2 \quad (22)$$

$$= -\gamma^2 \gamma^3 \gamma^2 \gamma^3 \quad (23)$$

$$= +\gamma^2 (\gamma^3)^2 \gamma^2 \quad (24)$$

$$= -(\gamma^2)^2 \quad (25)$$

$$= +1 \quad (26)$$

For  $v_{\pm}(\mathbf{0})$  we have, using 12

$$P_{\uparrow} v_{\pm}(\mathbf{0}) = \frac{1}{2} (I + \gamma^0 \sigma^{23}) v_{\pm}(\mathbf{0}) \quad (27)$$

$$= \frac{1}{2} (I + i\gamma^0 \gamma^2 \gamma^3) v_{\pm}(\mathbf{0}) \quad (28)$$

$$= \frac{1}{2} (I + i\gamma^2 \gamma^3 \gamma^0) v_{\pm}(\mathbf{0}) \quad (29)$$

$$= \frac{1}{2} (I - i\gamma^2 \gamma^3) v_{\pm}(\mathbf{0}) \quad (30)$$

$$= \frac{1}{2} (I - \Sigma_x) v_{\pm}(\mathbf{0}) \quad (31)$$

so

$$\Sigma_x P_{\uparrow} v_{\pm}(\mathbf{0}) = \Sigma_x \frac{1}{2} (I - \Sigma_x) v_{\pm}(\mathbf{0}) \quad (32)$$

$$= \frac{1}{2} \left( \Sigma_x - (\Sigma_x)^2 \right) v_{\pm}(\mathbf{0}) \quad (33)$$

$$= \frac{1}{2} (\Sigma_x - I) v_{\pm}(\mathbf{0}) \quad (34)$$

$$= -P_{\uparrow} v_{\pm}(\mathbf{0}) \quad (35)$$

Thus

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$$\Sigma_x P_{\uparrow} u_{\pm}(\mathbf{0}) = P_{\uparrow} u_{\pm}(\mathbf{0}) \quad (36)$$

$$\Sigma_x P_{\uparrow} v_{\pm}(\mathbf{0}) = -P_{\uparrow} v_{\pm}(\mathbf{0}) \quad (37)$$

for any representation of the  $\gamma^{\mu}$ .