SPIN PROJECTION OPERATORS IN THE DIRAC EQUATION FOR A PARTICLE AT REST

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.21.

In the last post, we saw that the spin projection operators projected out spin components for a particle at rest, but we used the explicit Dirac-Pauli representation of the γ^{μ}. It is possible to show the same thing generally, without using any particular reprsentation for the γ^{μ}. That is, we want to show that for the spinors representing a particle at rest, given by

$$u_{+}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \tag{1}$$

$$u_{-}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \tag{2}$$

$$v_{+}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{3}$$

$$v_{-}(\mathbf{0})=\sqrt{2m}\begin{bmatrix}0\\0\\-1\\0\end{bmatrix} \tag{4}$$

and a projection operator given by

$$P_{\uparrow}=\frac{1}{2}\left(I+\gamma^{0}\sigma^{23}\right) \tag{5}$$

we get

$$\Sigma_x P_\uparrow u_\pm(\mathbf{0}) = P_\uparrow u_\pm(\mathbf{0}) \tag{6}$$
$$\Sigma_x P_\uparrow v_\pm(\mathbf{0}) = -P_\uparrow v_\pm(\mathbf{0}) \tag{7}$$

To do this, we first note that $u_\pm(\mathbf{0})$ and $v_\pm(\mathbf{0})$ are eigenstates of γ^0. From the definition of these spinors, they satisfy the equations

$$\left(\not{p} - m\right) u_\pm(\mathbf{p}) = 0 \tag{8}$$
$$\left(\not{p} + m\right) v_\pm(\mathbf{p}) = 0 \tag{9}$$

With $\mathbf{p} = 0$, $\not{p} = \gamma^0 p_0 = m\gamma^0$ so we have

$$\left(\not{p} - m\right) u_\pm(\mathbf{0}) = \left(m\gamma^0 - m\right) u_\pm(\mathbf{0}) = 0 \tag{10}$$

so

$$\gamma^0 u_\pm(\mathbf{0}) = u_\pm(\mathbf{0}) \tag{11}$$

Similarly

$$\gamma^0 v_\pm(\mathbf{0}) = -v_\pm(\mathbf{0}) \tag{12}$$

Returning to 5, we have

$$P_\uparrow u_\pm(\mathbf{0}) = \frac{1}{2}\left(I + \gamma^0\sigma^{23}\right) u_\pm(\mathbf{0}) \tag{13}$$
$$= \frac{1}{2}\left(I + i\gamma^0\gamma^2\gamma^3\right) u_\pm(\mathbf{0}) \tag{14}$$
$$= \frac{1}{2}\left(I + i\gamma^2\gamma^3\gamma^0\right) u_\pm(\mathbf{0}) \tag{15}$$
$$= \frac{1}{2}\left(I + i\gamma^2\gamma^3\right) u_\pm(\mathbf{0}) \tag{16}$$
$$= \frac{1}{2}\left(I + \Sigma_x\right) u_\pm(\mathbf{0}) \tag{17}$$

We can now operate on this state with $\Sigma_x = i\gamma^2\gamma^3$ to get

$$\Sigma_x P_\uparrow u_\pm(\mathbf{0}) = \Sigma_x \frac{1}{2}(I+\Sigma_x)\, u_\pm(\mathbf{0}) \tag{18}$$
$$= \frac{1}{2}\left(\Sigma_x + (\Sigma_x)^2\right) u_\pm(\mathbf{0}) \tag{19}$$
$$= \frac{1}{2}(\Sigma_x + I)\, u_\pm(\mathbf{0}) \tag{20}$$
$$= P_\uparrow u_\pm(\mathbf{0}) \tag{21}$$

where we've used

$$(\Sigma_x)^2 = \left(i\gamma^2\gamma^3\right)^2 \tag{22}$$
$$= -\gamma^2\gamma^3\gamma^2\gamma^3 \tag{23}$$
$$= +\gamma^2\left(\gamma^3\right)^2\gamma^2 \tag{24}$$
$$= -\left(\gamma^2\right)^2 \tag{25}$$
$$= +1 \tag{26}$$

For $v_\pm(\mathbf{0})$ we have, using 12

$$P_\uparrow v_\pm(\mathbf{0}) = \frac{1}{2}\left(I+\gamma^0\sigma^{23}\right) v_\pm(\mathbf{0}) \tag{27}$$
$$= \frac{1}{2}\left(I+i\gamma^0\gamma^2\gamma^3\right) v_\pm(\mathbf{0}) \tag{28}$$
$$= \frac{1}{2}\left(I+i\gamma^2\gamma^3\gamma^0\right) v_\pm(\mathbf{0}) \tag{29}$$
$$= \frac{1}{2}\left(I-i\gamma^2\gamma^3\right) v_\pm(\mathbf{0}) \tag{30}$$
$$= \frac{1}{2}(I-\Sigma_x)\, v_\pm(\mathbf{0}) \tag{31}$$

so

$$\Sigma_x P_\uparrow v_\pm(\mathbf{0}) = \Sigma_x \frac{1}{2}(I-\Sigma_x)\, v_\pm(\mathbf{0}) \tag{32}$$
$$= \frac{1}{2}\left(\Sigma_x - (\Sigma_x)^2\right) v_\pm(\mathbf{0}) \tag{33}$$
$$= \frac{1}{2}(\Sigma_x - I)\, v_\pm(\mathbf{0}) \tag{34}$$
$$= -P_\uparrow v_\pm(\mathbf{0}) \tag{35}$$

Thus

$$\Sigma_x P_\uparrow u_\pm(\mathbf{0}) = P_\uparrow u_\pm(\mathbf{0}) \tag{36}$$

$$\Sigma_x P_\uparrow v_\pm(\mathbf{0}) = -P_\uparrow v_\pm(\mathbf{0}) \tag{37}$$

for any representation of the γ^μ.