LAGRANGIAN FOR THE DIRAC FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.22.

The Lagrangian that gives rise to the Dirac equation can be taken to be

$$\mathscr{L}=\overline{\psi}\left(i\partial\!\!\!/-m\right)\psi \tag{1}$$

As the spinors ψ and $\overline{\psi}$ are 4-component vectors, this is actually a matrix equation, although for the most part, we can treat ψ and $\overline{\psi}$ as though they were single functions when calculating the equations of motion from the Euler-Lagrange equations. The Lagrangian written out as an equation with explicit matrix elements is

$$\mathscr{L}=\overline{\psi}_{\alpha}\left(i\left(\gamma^{\mu}\right)_{\alpha\beta}\partial_{\mu}-m\delta_{\alpha\beta}\right)\psi_{\beta} \tag{2}$$

where α and β are the indexes for the vector and matrix components, and the γ^{μ} are the gamma matrices.

Since we treat ψ and $\overline{\psi}$ as two independent fields, we can derive the equations of motion by considering the Euler-Lagrange equation for each of these fields. Choosing $\overline{\psi}$, we have

$$\partial_{\mu}\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\overline{\psi}_{\alpha}\right)}\right)=\frac{\partial\mathscr{L}}{\partial\overline{\psi}_{\alpha}} \tag{3}$$

where the index α indicates which of the four components of $\overline{\psi}$ we're considering.

As there are no terms involving $\partial_{\mu}\overline{\psi}_{\alpha}$ in 1, the LHS is zero and we get from the RHS:

$$\frac{\partial\mathscr{L}}{\partial\overline{\psi}_{\alpha}}=\left(i\left(\gamma^{\mu}\right)_{\alpha\beta}\partial_{\mu}-m\delta_{\alpha\beta}\right)\psi_{\beta}=0 \tag{4}$$

There are four of these equations, one for each value of α. We can combine them into a single matrix equation, which we can see is just the Dirac equation for a free particle

$$\left(i\partial\!\!\!/ - m\right)\psi = 0 \tag{5}$$

We can also derive the Dirac equation by considering the Euler-Lagrange equation for the other field, ψ. The equation is now

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \psi_\beta\right)} \right) = \frac{\partial \mathscr{L}}{\partial \psi_\beta} \tag{6}$$

This time, the LHS is not zero, since there is a term involving $\partial_\mu \psi_\alpha$ in 1. From 2 we have

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \psi_\beta\right)} \right) = i\partial_\mu \left(\overline{\psi}_\alpha \left(\gamma^\mu\right)_{\alpha\beta} \right) \tag{7}$$

$$i\left(\overline{\psi}\overleftarrow{\partial\!\!\!/}\right)_\beta \tag{8}$$

where the left-pointing arrow indicates that the derivative operator acts on the object to its left.

The RHS of 6 gives us

$$\frac{\partial \mathscr{L}}{\partial \psi_\beta} = -m\overline{\psi}_\alpha \delta_{\alpha\beta} \tag{9}$$

$$= -m\left(\overline{\psi} I\right)_\beta \tag{10}$$

where I is the 4×4 identity matrix. We can combine 8 and 10 to get the matrix equation

$$\left[\overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right)\right] = 0 \tag{11}$$

This is the hermitian conjugate of the original Dirac equation 5. To see this, we use the identities

$$\left(\gamma^\mu\right)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \tag{12}$$

$$\left(\gamma^0\right)^2 = 1 \tag{13}$$

We have

$$\left[\overline{\psi}\left(i\overleftarrow{\not\partial}+m\right)\right]^{\dagger}=\left[\psi^{\dagger}\gamma^{0}\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu}+m\right)\right]^{\dagger} \tag{14}$$
$$=\left(-i\left(\gamma^{\mu}\right)^{\dagger}\partial_{\mu}+m\right)\left(\gamma^{0}\right)^{\dagger}\psi \tag{15}$$
$$=\left(-i\gamma^{0}\gamma^{\mu}\gamma^{0}\partial_{\mu}+m\right)\gamma^{0}\psi \tag{16}$$
$$=\left(-i\gamma^{0}\gamma^{\mu}\gamma^{0}\gamma^{0}\partial_{\mu}+\gamma^{0}m\right)\psi \tag{17}$$
$$=\gamma^{0}\left(-i\gamma^{\mu}\partial_{\mu}+m\right)\psi \tag{18}$$
$$=0 \tag{19}$$

We can now multiply from the left by $-\gamma^{0}$ to get

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=\left(i\not\partial-m\right)\psi=0 \tag{20}$$

which is the original Dirac equation 5.

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