

LAGRANGIAN FOR THE DIRAC FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.22.

The Lagrangian that gives rise to the Dirac equation can be taken to be

$$\mathcal{L} = \bar{\psi} (i\partial - m) \psi \quad (1)$$

As the spinors ψ and $\bar{\psi}$ are 4-component vectors, this is actually a matrix equation, although for the most part, we can treat ψ and $\bar{\psi}$ as though they were single functions when calculating the equations of motion from the Euler-Lagrange equations. The Lagrangian written out as an equation with explicit matrix elements is

$$\mathcal{L} = \bar{\psi}_\alpha \left(i(\gamma^\mu)_{\alpha\beta} \partial_\mu - m\delta_{\alpha\beta} \right) \psi_\beta \quad (2)$$

where α and β are the indexes for the vector and matrix components, and the γ^μ are the gamma matrices.

Since we treat ψ and $\bar{\psi}$ as two independent fields, we can derive the equations of motion by considering the Euler-Lagrange equation for each of these fields. Choosing $\bar{\psi}$, we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}_\alpha)} \right) = \frac{\partial \mathcal{L}}{\partial \bar{\psi}_\alpha} \quad (3)$$

where the index α indicates which of the four components of $\bar{\psi}$ we're considering.

As there are no terms involving $\partial_\mu \bar{\psi}_\alpha$ in 1, the LHS is zero and we get from the RHS:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_\alpha} = \left(i(\gamma^\mu)_{\alpha\beta} \partial_\mu - m\delta_{\alpha\beta} \right) \psi_\beta = 0 \quad (4)$$

There are four of these equations, one for each value of α . We can combine them into a single matrix equation, which we can see is just the Dirac equation for a free particle

$$(i\rlap{-}\not{\partial} - m)\psi = 0 \quad (5)$$

We can also derive the Dirac equation by considering the Euler-Lagrange equation for the other field, $\bar{\psi}$. The equation is now

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_\beta)} \right) = \frac{\partial \mathcal{L}}{\partial \psi_\beta} \quad (6)$$

This time, the LHS is not zero, since there is a term involving $\partial_\mu \psi_\alpha$ in 1. From 2 we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_\beta)} \right) = i\partial_\mu \left(\bar{\psi}_\alpha (\gamma^\mu)_{\alpha\beta} \right) \quad (7)$$

$$i \left(\bar{\psi} \overleftarrow{\not{\partial}} \right)_\beta \quad (8)$$

where the left-pointing arrow indicates that the derivative operator acts on the object to its left.

The RHS of 6 gives us

$$\frac{\partial \mathcal{L}}{\partial \psi_\beta} = -m \bar{\psi}_\alpha \delta_{\alpha\beta} \quad (9)$$

$$= -m (\bar{\psi} I)_\beta \quad (10)$$

where I is the 4×4 identity matrix. We can combine 8 and 10 to get the matrix equation

$$\left[\bar{\psi} \left(i \overleftarrow{\not{\partial}} + m \right) \right] = 0 \quad (11)$$

This is the hermitian conjugate of the original Dirac equation 5. To see this, we use the identities

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (12)$$

$$(\gamma^0)^2 = 1 \quad (13)$$

We have

$$\left[\bar{\psi} \left(i \overleftarrow{\not{\partial}} + m \right)\right]^\dagger = \left[\psi^\dagger \gamma^0 \left(i \gamma^\mu \overleftarrow{\partial}_\mu + m \right)\right]^\dagger \quad (14)$$

$$= \left(-i (\gamma^\mu)^\dagger \partial_\mu + m \right) (\gamma^0)^\dagger \psi \quad (15)$$

$$= \left(-i \gamma^0 \gamma^\mu \gamma^0 \partial_\mu + m \right) \gamma^0 \psi \quad (16)$$

$$= \left(-i \gamma^0 \gamma^\mu \gamma^0 \gamma^0 \partial_\mu + \gamma^0 m \right) \psi \quad (17)$$

$$= \gamma^0 \left(-i \gamma^\mu \partial_\mu + m \right) \psi \quad (18)$$

$$= 0 \quad (19)$$

We can now multiply from the left by $-\gamma^0$ to get

$$(i \gamma^\mu \partial_\mu - m) \psi = (i \not{\partial} - m) \psi = 0 \quad (20)$$

which is the original Dirac equation 5.

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