HERMITIAN LAGRANGIAN FOR THE DIRAC FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.23.

The Lagrangian that gives rise to the Dirac equation can be taken to be

$$\mathscr{L}=\overline{\psi}\left(i\partial\!\!\!/-m\right)\psi \tag{1}$$

As the spinors ψ and $\overline{\psi}$ are 4-component vectors, this is actually a matrix equation. As pointed out by L&P in their equation 4.88, this Lagrangian is not hermitian. It is possible to use a hermitian Lagrangian of the form

$\overline{\psi}$ is a row vector with 4 components and ψ is a column vector with 4 components.

$$\mathscr{L}^{\prime}=\frac{i}{2}\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi-\frac{i}{2}\left(\partial_{\mu}\overline{\psi}\right)\gamma^{\mu}\psi-m\overline{\psi}\psi \tag{2}$$

L&P also show that this Lagrangian differs from 1 by a total divergence of form

$$\mathscr{L}-\mathscr{L}^{\prime}=\partial_{\mu}\left(\frac{i}{2}\overline{\psi}\gamma^{\mu}\psi\right) \tag{3}$$

and by the analysis given in their equations 2.24 and 2.25, the variations in the actions obtained by these two Lagrangians are the same, so the equations of motion are also the same. We can see that explicitly by calculating the Euler-Lagrange equations for 2. These equations are, for $\overline{\psi}$:

$$\partial_{\mu}\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\overline{\psi}_{\alpha}\right)}\right)=\frac{\partial\mathscr{L}}{\partial\overline{\psi}_{\alpha}} \tag{4}$$

where the subscript α denotes one of the four components of $\overline{\psi}$. We can write out 2 in terms of components as

$$\mathscr{L}^{\prime}=\frac{i}{2}\overline{\psi}_{\alpha}\left(\gamma^{\mu}\right)_{\alpha\beta}\partial_{\mu}\psi_{\beta}-\frac{i}{2}\left(\partial_{\mu}\overline{\psi}_{\alpha}\right)\left(\gamma^{\mu}\right)_{\alpha\beta}\psi_{\beta}-m\overline{\psi}_{\alpha}I_{\alpha\beta}\psi_{\beta} \tag{5}$$

where I is the 4×4 identity matrix.

From 4 we have for the LHS

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\overline{\psi}_\alpha\right)}\right) = -\frac{i}{2}\partial_\mu\left(\left(\gamma^\mu\right)_{\alpha\beta}\psi_\beta\right) \tag{6}$$

$$= -\frac{i}{2}\left(\gamma^\mu\right)_{\alpha\beta}\partial_\mu\psi_\beta \tag{7}$$

The RHS gives

$$\frac{\partial\mathscr{L}}{\partial\overline{\psi}_\alpha} = \frac{i}{2}\left(\gamma^\mu\right)_{\alpha\beta}\partial_\mu\psi_\beta - mI_{\alpha\beta}\psi_\beta \tag{8}$$

Putting them together gives

$$i\left(\gamma^\mu\right)_{\alpha\beta}\partial_\mu\psi_\beta - mI_{\alpha\beta}\psi_\beta = 0 \tag{9}$$

or, using the slash notation for the full matrix equation

$$\left(i\not\partial - m\right)\psi = 0 \tag{10}$$

which is the original Dirac equation.

If we work out the Euler-Lagrange equation for ψ, we get

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\psi_\beta\right)}\right) = \frac{i}{2}\partial_\mu\left(\overline{\psi}_\alpha\right)\left(\gamma^\mu\right)_{\alpha\beta} \tag{11}$$

$$\frac{\partial\mathscr{L}}{\partial\psi_\beta} = -\frac{i}{2}\left(\partial_\mu\overline{\psi}_\alpha\right)\left(\gamma^\mu\right)_{\alpha\beta} - m\overline{\psi}_\alpha I_{\alpha\beta} \tag{12}$$

Combining them we get

$$i\partial_\mu\left(\overline{\psi}_\alpha\right)\left(\gamma^\mu\right)_{\alpha\beta} + m\overline{\psi}_\alpha I_{\alpha\beta} = 0 \tag{13}$$

or

$$\left[\overline{\psi}\left(i\overleftarrow{\not\partial} + m\right)\right] = 0 \tag{14}$$

where the left arrow means that we take the derivative of $\overline{\psi}$ to is left. This is just the hermitian conjugate equation we met earlier.

The Lagrangian 5 is invariant under the phase change

$$\psi \to e^{-iq\theta}\psi \tag{15}$$

where θ and q are parameters. According to Noether's theorem, this internal symmetry gives rise to a conserved current, which is

$$J^\mu = \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\frac{\delta\Phi^A}{\delta\varepsilon} \tag{16}$$

where the Φ^As are the independent fields, which in our case are ψ and $\overline{\psi}$, and $\delta\varepsilon$ is the infinitesimal change in the parameter, in our case θ. The infinitesimal change in the field is

$$\delta\psi_\beta = -iq\theta\psi_\beta \tag{17}$$
$$\delta\overline{\psi}_\alpha = +iq\theta\overline{\psi}_\alpha \tag{18}$$

so we get

$$J^\mu = \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\overline{\psi}_\alpha\right)} iq\overline{\psi}_\alpha - \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\psi_\beta\right)} iq\psi_\beta \tag{19}$$
$$= -\frac{i}{2}\left(\gamma^\mu\right)_{\alpha\beta}\psi_\beta iq\overline{\psi}_\alpha - \frac{i}{2}\overline{\psi}_\alpha\left(\gamma^\mu\right)_{\alpha\beta} iq\psi_\beta \tag{20}$$
$$= \overline{\psi}_\alpha\left(\gamma^\mu\right)_{\alpha\beta} q\psi_\beta \tag{21}$$
$$= q\overline{\psi}\gamma^\mu\psi \tag{22}$$

If we tried to do this without using matrix indexes α and β, we would get the first expression with a column vector ψ multiplied by a row vector $\overline{\psi}$ (which is a 4×4 matrix), and a second expression with $\overline{\psi}$ multiplied by ψ, which is a scalar. Putting in the indexes allows us to get the matrix multiplication correct.

The conserved charge is given by

$$Q = \int d^3x\, J^0 \tag{23}$$
$$= q\int d^3x\, \overline{\psi}\gamma^0\psi \tag{24}$$
$$= q\int d^3x\, \psi^\dagger\gamma^0\gamma^0\psi \tag{25}$$
$$= q\int d^3x\, \psi^\dagger\psi \tag{26}$$

which is the same as that obtained from applying Noether's theorem to the original Lagrangian 1, as given in L&P's equation 4.93.