PAULI-LUBANSKY VECTOR AND SPIN IN THE DIRAC EQUATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.24.

The Pauli-Lubansky vector is defined by

$$W_\mu \equiv -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}P^\nu J^{\lambda\rho} \tag{1}$$

where P^ν is the momentum operator and $J^{\lambda\rho}$ is the angular momentum operator. There is a theorem (quoted by L&P so I don't know its derivation) that states

$$W^\mu W_\mu = -m^2 s(s+1) \tag{2}$$

where s is the spin of the particle. The problem here is to use the form of $J^{\lambda\rho}$ derived earlier to show that particle satisfying the Dirac equation has spin $\frac{1}{2}$. The expression for angular momentum is

$$J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} \tag{3}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{4}$$

At first glance, the calculation of the LHS of 2 looks extrememly complicated, as it involves the contraction of the antisymmetric tensors $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\alpha\beta\gamma}$ and the product of four factors. The contraction formula is given in L&P's Appendix A.3 as formula A.32, which is (with L&P's g replaced by the Kronecker delta, since that's what it is, really, and indices modified to suit our problem):

$$\begin{aligned}\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\alpha\beta\gamma} = -\Big(&\delta_\nu^\alpha\delta_\lambda^\beta\delta_\rho^\gamma + \delta_\lambda^\alpha\delta_\rho^\beta\delta_\nu^\gamma + \delta_\rho^\alpha\delta_\nu^\beta\delta_\lambda^\gamma \\ &-\delta_\lambda^\alpha\delta_\nu^\beta\delta_\rho^\gamma - \delta_\rho^\alpha\delta_\lambda^\beta\delta_\nu^\gamma - \delta_\nu^\alpha\delta_\rho^\beta\delta_\lambda^\gamma\Big)\end{aligned} \tag{5}$$

Substituting 1 into 2 and expanding the contraction, we get

$$W^\mu W_\mu = -\frac{1}{4} P_\alpha J_{\beta\gamma} \left(P^\alpha J^{\beta\gamma} + P^\gamma J^{\alpha\beta} + P^\beta J^{\gamma\alpha} - P^\beta J^{\alpha\gamma} - P^\gamma J^{\beta\alpha} - P^\alpha J^{\gamma\beta} \right) \tag{6}$$

From 3 and the property of $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$, we see that $J^{\mu\nu} = -J^{\nu\mu}$, so the last 3 terms in 6 are equal to the first 3, so we have

$$W^\mu W_\mu = -\frac{1}{2} P_\alpha J_{\beta\gamma} \left(P^\alpha J^{\beta\gamma} + P^\gamma J^{\alpha\beta} + P^\beta J^{\gamma\alpha} \right) \tag{7}$$

Although this looks a bit simpler, it is still complicated to multiply out all the terms on the RHS after substituting for $J^{\mu\nu}$ using 3. However, I think the key is to notice that the quantity $W^\mu W_\mu$ is a scalar, and thus has the same value in any inertial frame. As such, we can calculate it in the rest frame of the particle, where

$$P^\mu = (m, 0, 0, 0) \tag{8}$$

In the rest frame, the orbital angular momentum component of $J^{\mu\nu}$ will also be zero, so it simplifies to

$$J_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} \tag{9}$$

Using these simplifications, we can now write 7 as

$$\begin{aligned} W^\mu W_\mu &= -\frac{1}{2} m^2 \left(J_{\beta\gamma} J^{\beta\gamma} + J_{\beta 0} J^{0\beta} + J_{0\beta} J^{\beta 0} \right) && (10) \\ &= -\frac{1}{8} m^2 \left(\sigma_{\beta\gamma} \sigma^{\beta\gamma} + \sigma_{\beta 0} \sigma^{0\beta} + \sigma_{0\beta} \sigma^{\beta 0} \right) && (11) \\ &= -\frac{1}{8} m^2 \left(\sigma_{\beta\gamma} \sigma^{\beta\gamma} - 2\sigma_{0\beta} \sigma^{0\beta} \right) && (12) \end{aligned}$$

where we used $\sigma_{0\beta} = -\sigma_{\beta 0}$ to get the last line. We can now work out the terms in the parentheses. We use the definition of $\sigma_{\mu\nu}$ and the properties of the γ^μ:

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{i}{2} \left[\gamma_\mu, \gamma_\nu \right] && (13) \\ &= \frac{i}{2} \left(\gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\nu - 2g_{\mu\nu} \right) && (14) \\ &= i \left(\gamma_\mu \gamma_\nu - g_{\mu\nu} \right) && (15) \end{aligned}$$

$$\sigma_{\beta\gamma}\sigma^{\beta\gamma} = i^2\left(\gamma_\beta\gamma_\gamma - g_{\beta\gamma}\right)\left(\gamma^\beta\gamma^\gamma - g^{\beta\gamma}\right) \tag{16}$$

$$= -\gamma_\beta\gamma_\gamma\gamma^\beta\gamma^\gamma + g_{\beta\gamma}\gamma^\beta\gamma^\gamma + g^{\beta\gamma}\gamma_\beta\gamma_\gamma - g_{\beta\gamma}g^{\beta\gamma} \tag{17}$$

$$= -\gamma_\beta\gamma_\gamma\left(-\gamma^\gamma\gamma^\beta + 2g^{\gamma\beta}\right) + 2g^{\beta\gamma}\gamma_\beta\gamma_\gamma - g_{\beta\gamma}g^{\beta\gamma} \tag{18}$$

$$= 16 - 8 + 8 - 4 \tag{19}$$

$$= 12 \tag{20}$$

where we've used the contraction formula

$$\gamma_\mu\gamma^\mu = 4 \tag{21}$$

For the second term in 12 we have, since $\sigma_{00} = 0$ and $(\sigma_{0i})^2 = 1$

$$\sigma_{0\beta}\sigma^{0\beta} = \sigma_{0i}\sigma^{0i} = 3 \tag{22}$$

Putting all this into 12 we have

$$W^\mu W_\mu = -\frac{1}{8}m^2(12-6) \tag{23}$$

$$= -\frac{3}{4}m^2 \tag{24}$$

Comparing with 2 we see that this gives

$$s = \frac{1}{2} \tag{25}$$

[I'm hoping that my assumption that we can work this out in the rest frame is valid, but if anyone has actually worked through the whole thing for arbitrary P^μ and the full form 3 for $J^{\mu\nu}$, please do let me know.]