ANTICOMMUTATION RELATIONS AND HAMILTONIAN FOR DIRAC FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.25.

As with the complex scalar field, we can quantize the Dirac field by introducing creation and annihilation operators as coefficients in a Fourier expansion over the plane wave solutions. The expansions are

$$\psi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + \hat{f}_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ip\cdot x} \right) \quad (1)$$

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s^\dagger(\mathbf{p}) \overline{u}_s(\mathbf{p}) e^{ip\cdot x} + \hat{f}_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) e^{-ip\cdot x} \right) \quad (2)$$

The u_s and v_s are the 4-component spinors we met earlier.

L&P show in equations 4.98 to 4.102 that the Dirac hamiltonian can be written in terms of the creation and annihilation operators as

$$H = \int d^3p\, E_p \sum_{s=1,2} \left[f_s^\dagger(\mathbf{p}) f_s(\mathbf{p}) - \hat{f}_s(\mathbf{p}) \hat{f}_s^\dagger(\mathbf{p}) \right] \quad (3)$$

Because of the minus sign before the second term, the Hamiltonian can take on both positive and negative values. We ran across a related problem (of infinite energy) with scalar fields, and there we 'fixed' the problem by imposing normal ordering on the operator products, in which creation operators were always placed to the left of annihilation operators. In the Dirac hamiltonian, this won't work since, even if we reversed the order of $\hat{f}_s$ and $\hat{f}_s^\dagger$ in the last term of 3, we still have the minus sign which will allow negative energies. The solution here is to impose anticommutation relations (as opposed to commutation relations for scalar field) on the creation and annihilation operators. That is, we require

$$\left\{ f_r(\mathbf{p}), f_s^\dagger(\mathbf{q}) \right\} = \left\{ \hat{f}_r(\mathbf{p}), \hat{f}_s^\dagger(\mathbf{q}) \right\} = \delta_{rs} \delta^3(\mathbf{p} - \mathbf{q}) \quad (4)$$

with all other anticommutators being zero. In this case

$$\hat{f}_s(\mathbf{p})\,\hat{f}_s^\dagger(\mathbf{p}) = -\hat{f}_s^\dagger(\mathbf{p})\,\hat{f}_s(\mathbf{p}) \tag{5}$$

so the Hamiltonian becomes

$$:H := \int d^3p\,E_p \sum_{s=1,2}\left[f_s^\dagger(\mathbf{p})\,f_s(\mathbf{p}) - \hat{f}_s^\dagger(\mathbf{p})\,\hat{f}_s(\mathbf{p})\right] \tag{6}$$

where the colons surrounding H denote normal ordering.

We can use the anticommutators 4 to derive anticommutators for the Dirac fields ψ and $\psi^\dagger$. It turns out to be easier to consider $\{\psi_a(t,\mathbf{x}),\overline{\psi}_b(t,\mathbf{y})\}$ first. Note that it's important to consider the relations between individual components of ψ and $\overline{\psi}$. If we tried to calculate $\{\psi(t,\mathbf{x}),\overline{\psi}(t,\mathbf{y})\}$ directly, the expression makes no sense, since ψ is a 4-component column vector and $\overline{\psi}$ is a 4-component row vector, so we can't add $\psi\overline{\psi}$ (which is a 4×4 matrix) to $\overline{\psi}\psi$ (which is a scalar).

Using 1 and 2 to calculate $\{\psi_a(\mathbf{x}),\overline{\psi}_b(\mathbf{y})\}$ can get messy, so to simplify the notation, I'll use

$$f_{sp} \equiv f_s(\mathbf{p}) \tag{7}$$

$$u_{spa} \equiv (u_s(\mathbf{p}))_a \tag{8}$$

with similar notation for $\hat{f}$ and v. The subscript a on u_{spa} indicates the a-th component of $u_s(\mathbf{p})$, so a can be 0, 1, 2 or 3.

Now for the calculation. We have

$$\{\psi_a(t,\mathbf{x}),\overline{\psi}_b(t,\mathbf{y})\} = \frac{1}{2(2\pi)^3}\left\{\int \frac{d^3p}{\sqrt{E_p}}\sum_{r=1,2}\left(f_{rp}u_{rpa}e^{-ip\cdot x} + \hat{f}_{rp}^\dagger v_{rpa}e^{ip\cdot x}\right),\right. \tag{9}$$

$$\left.\int \frac{d^3q}{\sqrt{E_q}}\sum_{s=1,2}\left(f_{sq}^\dagger \overline{u}_{sqb}e^{iq\cdot y} + \hat{f}_{sq}\overline{v}_{sqb}e^{-iq\cdot y}\right)\right\} \tag{10}$$

In this notation, every term except for the f and $\hat{f}$ operators is just a number, so these terms commute with everything. Because all anticommutators except 4 are zero, the only anticommutators we need to consider are $\left\{f_{rp},f_{sq}^\dagger\right\}$ and $\left\{\hat{f}_{rp}^\dagger,\hat{f}_{sq}\right\}$. We therefore have

$$\left\{\psi_a(t,\mathbf{x}),\overline{\psi}_b(t,\mathbf{y})\right\}=\frac{1}{2(2\pi)^3}\int\frac{d^3p\,d^3q}{\sqrt{E_pE_q}}\sum_{r,s}\left(\left\{f_{rp},f_{sq}^\dagger\right\}u_{rpa}\overline{u}_{sqb}e^{-i(p\cdot x-q\cdot y)}+\right. \tag{11}$$

$$\left.\left\{\hat{f}_{rp}^\dagger,\hat{f}_{sq}\right\}v_{rpa}\overline{v}_{sqb}e^{i(p\cdot x-q\cdot y)}\right) \tag{12}$$

We now have

$$\left\{f_{rp},f_{sq}^\dagger\right\}=\left\{\hat{f}_{rp}^\dagger,\hat{f}_{sq}\right\}=\delta_{rs}\delta^3(\mathbf{p}-\mathbf{q}) \tag{13}$$

We can use this to do one of the sums and one of the integrals, leaving us with

$$\left\{\psi_a(t,\mathbf{x}),\overline{\psi}_b(t,\mathbf{y})\right\}=\frac{1}{2(2\pi)^3}\int\frac{d^3p}{E_p}\sum_s\left(u_{spa}\overline{u}_{spb}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\right. \tag{14}$$

$$\left.v_{spa}\overline{v}_{spb}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) \tag{15}$$

Although the integral is over the 3 space components of the momentum $\mathbf{q}$, the delta function sets $\mathbf{q}=\mathbf{p}$, which results in $q_0=E_q=E_p=p_0$ as well.

Because we're considering equal times in ψ_a and $\overline{\psi}_b$, the $p^0(x_0-y_0)$ term in the exponentials is zero, so the exponents now contain scalar products of 3-vectors only.

We can now do the sums using the identities

$$\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p})=\not{p}+m \tag{16}$$

$$\sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p})=\not{p}-m \tag{17}$$

In terms of components, these identities are

$$\left[\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p})\right]_{ab}=\sum_s u_{spa}\overline{u}_{spb}=\left[\not{p}+m\right]_{ab} \tag{18}$$

$$\left[\sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p})\right]_{ab}=\sum_s v_{spa}\overline{v}_{spb}=\left[\not{p}-m\right]_{ab} \tag{19}$$

We can therefore regard the LHS of 14 as a 4×4 matrix with components $\left\{\psi(\mathbf{x}),\overline{\psi}(\mathbf{y})\right\}_{ab}$ and therefore write 14 as a matrix equation:

$$\left\{\psi(t,\mathbf{x}),\overline{\psi}(t,\mathbf{y})\right\} = \frac{1}{2(2\pi)^3}\int\frac{d^3p}{E_p}\left[\left(\not p+m\right)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left(\not p-m\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right] \tag{20}$$

$$= \frac{1}{2(2\pi)^3}\int\frac{d^3p}{E_p}\left[\left(p_0\gamma^0+\mathbf{p}\cdot\gamma+m\right)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left(p_0\gamma^0+\mathbf{p}\cdot\gamma-m\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right] \tag{21}$$

We can now change the variable of integration from $\mathbf{p}\to-\mathbf{p}$ in the second term. This causes $d^3p\to-d^3p$ but the three limits of integration also change direction so the net result is still an integration over d^3p from $-\infty$ to $+\infty$ in all 3 components. We then get

$$\left\{\psi(t,\mathbf{x}),\overline{\psi}(t,\mathbf{y})\right\} = \frac{1}{2(2\pi)^3}\int\frac{d^3p}{E_p}\left[\left(p_0\gamma^0+\mathbf{p}\cdot\gamma+m+p_0\gamma^0-\mathbf{p}\cdot\gamma-m\right)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right] \tag{22}$$

$$= \frac{1}{(2\pi)^3}\int\frac{d^3p}{E_p}p_0\gamma^0e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{23}$$

$$= \frac{1}{(2\pi)^3}\int d^3pe^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\gamma^0 \tag{24}$$

$$= \delta^3(\mathbf{x}-\mathbf{y})\gamma^0 \tag{25}$$

where we used $p_0=E_p$ to get the third line. We can now multiply by γ^0 on both sides and use $\left(\gamma^0\right)^2=1$ to get the final result:

$$\left\{\psi(t,\mathbf{x}),\psi^\dagger(t,\mathbf{y})\right\} = \delta^3(\mathbf{x}-\mathbf{y}) \tag{26}$$

This should be interpreted as a matrix equation, where the LHS is $\left\{\psi_a(t,\mathbf{x}),\psi_b^\dagger(t,\mathbf{y})\right\}$ and the RHS is $\delta^3(\mathbf{x}-\mathbf{y})I_{ab}$. We can write this in terms of components as

$$\left\{\psi_a(t,\mathbf{x}),\psi_b^\dagger(t,\mathbf{y})\right\} = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) \tag{27}$$

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